Recall: Understood representations of $n = x^2 + y^2$ through $\mathbb{Z}[i]$, where $i^4 = 1$. "Review": what happens for ω: solution to $x^3 = 1$? Look at $\underbrace{x^4 - 1}$

$x^4 = 1$.

$e^{\frac{2\pi i}{4}}$

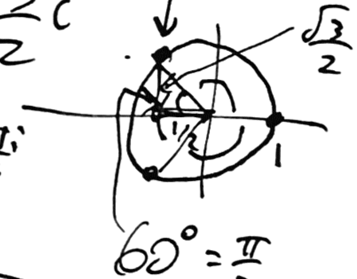
$= (x^2-1)(x^2+1)$

$= (x+1)(x-1) \cdots$

$$0 = x^3 - 1 \Rightarrow x = 1, \; x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$= (x-1)(x^2+x+1)$$

$$x = e^{0 \cdot \frac{2\pi i}{3}} \qquad x = e^{\pm \frac{2\pi i}{3}}$$



$60° = \frac{\pi}{3}$

$$N\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = \frac{1}{4} + \frac{3}{4} = 1$$

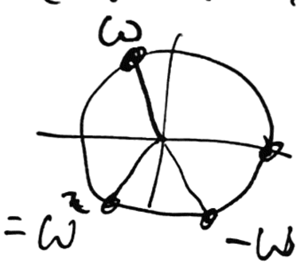
$$\boxed{e^{2\pi i} = 1}$$

Exercise 1: $\mathbb{Z}[\omega]$ is a ring, where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i = e^{\frac{2\pi i}{3}}$.

$$= \{a_0 + a_1 \omega + a_2 \omega^2 + a_3 \omega^3 + \dots + a_n \omega^n ; \; a_j \in \mathbb{Z}\} = \{a_0 + a_1 \omega\}.$$

Because ω solves $x^2 + x + 1 = 0$, so $\omega^2 = -\omega - 1$

$$\mathbb{Z}[\omega] = \mathbb{Z}[x]/(x^2+x+1).$$

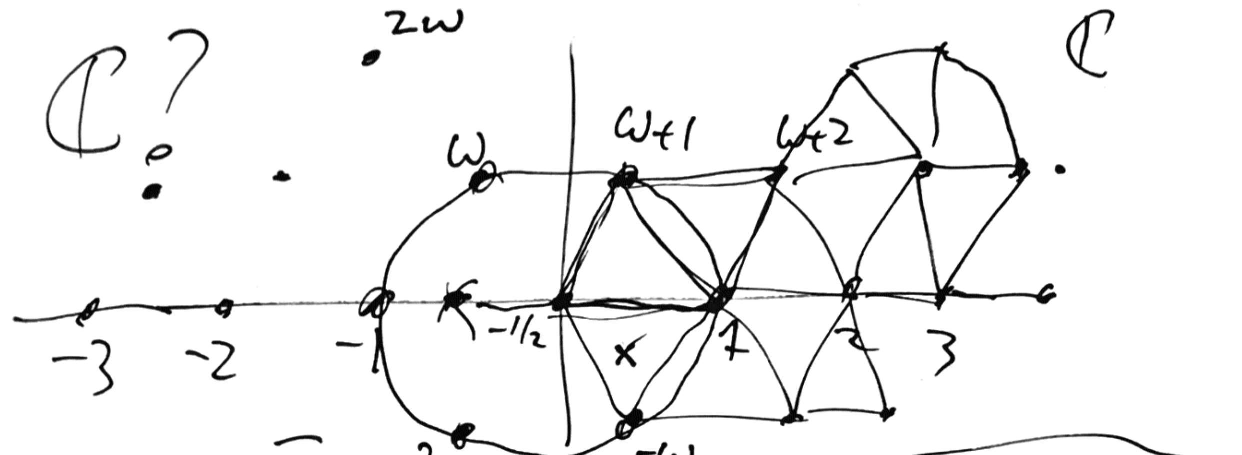


Q: What is $\mathbb{Z}[\omega] \subset \mathbb{C}$?

$\ni a+b\omega,\ a,b\in\mathbb{Z}.$

"hexagonal lattice".

"Eisenstein integers".



$\bar{\omega} = \omega^2 = -1-\omega$

Exercise 2: Prove

$N(a+b\omega) = (a+b\omega)(a+b\bar{\omega}) = a^2 - ab + b^2.$

Q: What is "norm form" for $\mathbb{Z}[\omega]$?

$$\omega^2 = \left(e^{\frac{2\pi i}{3}}\right)^2 = e^{\frac{4\pi i}{3}} = \cos\tfrac{4\pi}{3} + i\sin\tfrac{4\pi}{3} = -\tfrac{1}{2} + i\left(-\tfrac{\sqrt{3}}{2}\right)$$

Main Question: What $n\in\mathbb{Z}$ are represented by $Q(x,y) = x^2 - xy + y^2$?

A: $(a+b\omega)(a+b\bar{\omega}) = a^2 + b^2 + ab(\underbrace{\omega+\bar{\omega}}_{\omega+(-1-\omega)})$

$\omega\cdot\bar{\omega} = |\omega|^2$

Q: Do we have a Division Algorithm?

Exercise 3: Prove that $\forall n, m \in \mathbb{Z}[\omega]$, $m\neq 0$, $\exists\, q, r$ s.t. $n = mq + r$ & $N(r) < N(m)$.