Thm: $x^4+y^4=z^2$ in $\mathbb{Z} \Rightarrow xyz=0$.

Pf. If not, $(x^2)^2+(y^2)^2=z^2$ is a Pyth triple. If $\gcd(x,y,z)=d$, then $\left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d^2}\right)$ is pairwise primitive.

Proof: Exercise 1.

Assume (x,y,z) pairwise primitive, y even

$\Longrightarrow$ (param) $\exists$ coprime $(r,s)=1$ opposite parity with

$x^2=r^2-s^2$, $\boxed{y^2=2rs}$ $z=r^2+s^2$.

$x^2+s^2=r^2 \Rightarrow (x,s,r)$ prim. Pyth triple.

s even

$\Rightarrow$ (param) $\exists\ (u,v)=1$, opp par., s.t. $x=u^2-v^2$, $\underline{s=2uv}$, $\underline{r=u^2+v^2}$

coprime.

$y^2 = 2rs = 2(u^2+v^2)\cdot 2uv$ $\boxed{\cdot\ y=2y_1}$

$4y_1^2 = 4\,u\cdot v\,(u^2+v^2)$

perfect square

pairwise coprime.

$\Rightarrow u=a^2,\ v=b^2,\ u^2+v^2=c^2=r$

$c^2 = a^4+b^4$. $\boxed{\text{Claim: If } xyz\neq 0, \text{ then } c<z}$

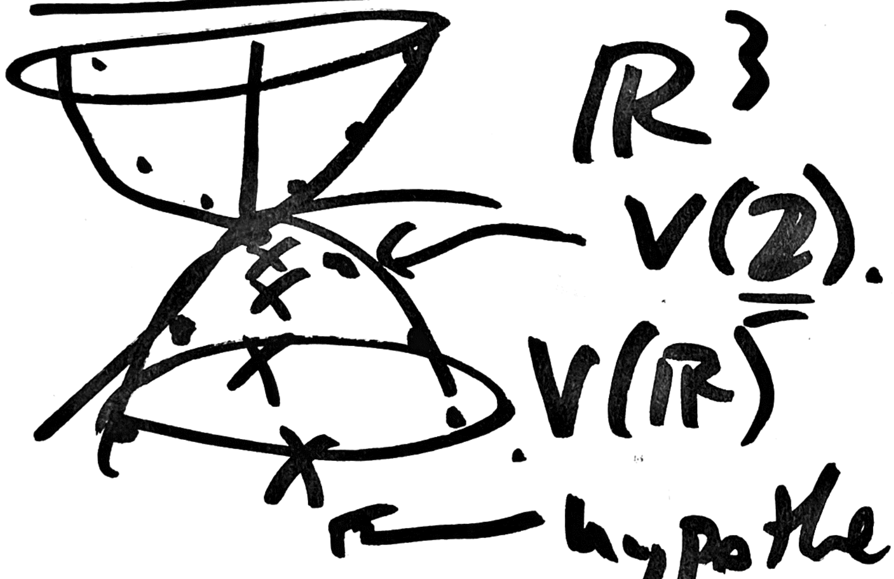
$\boxed{\text{Exercise 2}} \rightarrow$

$\boxed{\text{Exercise 3:}}$ Set (x,y,z)

What are: $r, s, u, \cancel{x}$ a, b, c? $\in (2,0,4)$.

So: if $\exists (x,y,z)$ with $xyz \neq 0$ & $x^4+y^4=z^2$, primitive $\Rightarrow$
$\exists (a,b,c)$ primitive with
$a^4+b^4=c^2$ & $abc \neq 0$ & $|c|<|z|$.

$\mathbb{R}^3$ $\quad V: x^4+y^4-z^2$.



Fermat's Descent idea: <u>continue</u> until $c=0$. $\Rightarrow a=0=b$. $\Rightarrow$ always had trivial solutions.

$\Rightarrow$ No nontrivial solutions.

$\Rightarrow$ No " solns to $x^4+y^4=z^4$.

Back to $x^2+y^2=z^2$.

$\Rightarrow x=r^2-s^2,\ y=2rs,\ \underline{z=r^2+s^2}$.

Q: What z's arise? I.e.

what numbers occur as hypotenuses of right $\mathbb{Z}$-Δ's.

I.e. let $Q(r,s)=\underbrace{r^2+s^2}$. Q is

an example of a quadratic <u>form</u>.

(Another quadratic form $Q_1(r,s)=3r^2-2rs+10s^2$)

(Another $Q_2(r,s)=r\cdot s$.)

<u>Another</u>: ~~wat~~ quad form: $Q_3(r,s,t,u)=\pi r^2+\sqrt{2}st+u^2$.

What whole numbers are represented as $Q(r,s)$? $r^2+s^2=z$?

homework: Make a table.

$s^2 \backslash r^2$	1	4	9	16	25	36	…
1	2	5	10	17	26	37	…
4	5	8	13	20	29	40	
9	10		18	$\boxed{25}$	34	45	…
16	17						
25	26						

What do we see?

$$z = 0, 1, 2, 4, 5, 8, 9, 10, 13, \ldots$$

homework: mess around, what do you notice???...