Recall: $V: x^2+y^2-z^2 \leftarrow$ polynomial

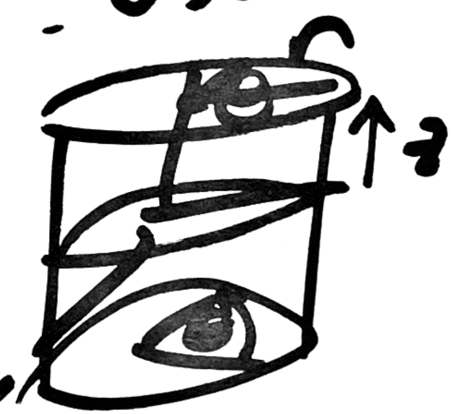
Variety (zero set). Look at $V_k: x^2+y^2-z^2-k$.

$V(\mathbb{R})$ = real points $\supset V(\mathbb{Z})$ = integer pts.

$W: x^2+y^2+z^2+1$. $W(\mathbb{R}) = \emptyset = W(\mathbb{Z})$.

$W(\mathbb{C}) \neq \emptyset$. real surface in $\mathbb{R}^3$.

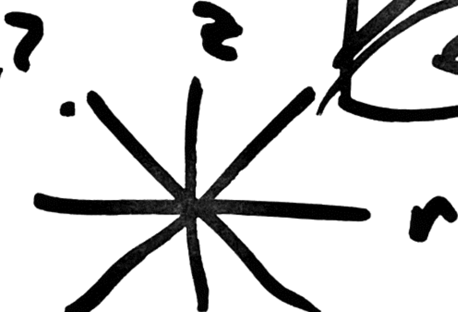
$\{\vec{v} \in V(\mathbb{R})\}$ radial in (x,y)-plane. Use cylindrical coords. $(r\cos\theta, r\sin\theta, z)$



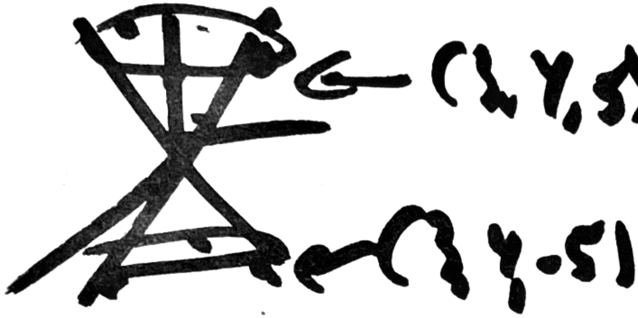
$V(\mathbb{R})$

$\Leftrightarrow r^2\cos^2\theta + r^2\sin^2\theta = r^2 = z^2$.

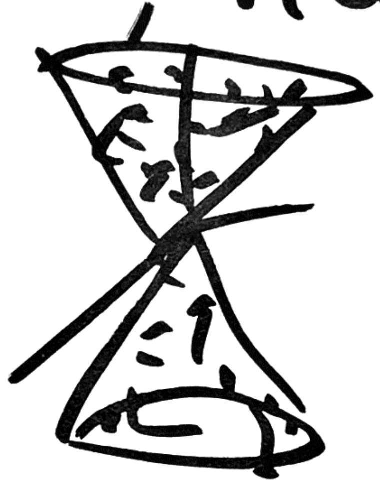
$\Rightarrow r = \pm z$.



$V(\mathbb{R})$:



$V(\mathbb{Z})$ $V(\mathbb{Q})$.



<u>Lemma</u>: $\forall \vec{v} \in V(\mathbb{Q})$

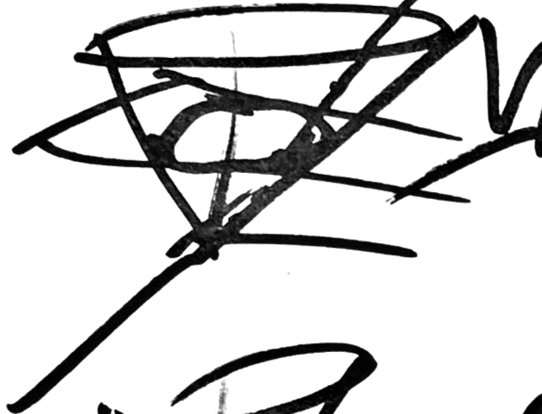
$\exists! \, \vec{w} \in V(\mathbb{Z})^{\text{or}}_{+}$ primitive with $\vec{w} \sim \vec{v}$. $z > 0$.

where $\vec{u} \sim \vec{v} \iff \vec{u} = \lambda \vec{v}, \; \lambda \in \mathbb{R} \setminus \{0\}$.

$\boxed{\text{Exercise 1}}$ Write out a complete proof.

$\boxed{\text{Exercise 0:}}$ Verify that $\sim$ is an equivalence relation.

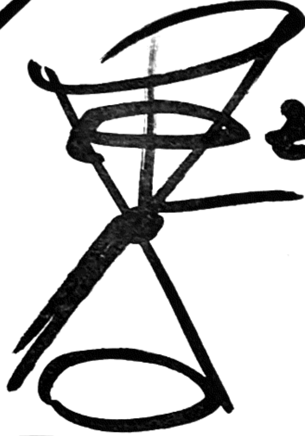
Q: What is $V(\mathbb{R})_H/\sim$? $V(\mathbb{R})/\sim$?



$V(\mathbb{R})_H \cap \{z=1\}$.

$$\underline{V(\mathbb{R})/\sim}\ \boxed{= S^1 \sqcup \{0\}.}$$

$$(x,y,\underset{\neq 0}{z}) \in V(\mathbb{R})_+ \rightsquigarrow \left(\frac{x}{z},\frac{y}{z},1\right) \in V(\mathbb{R})_+.$$



$z=1$. $(x,y,z) \xrightarrow{\text{scale by } \frac{1}{z}} \frac{1}{z}\ \left(\frac{x}{z},\frac{y}{z},1\right)$.

unless $z=0$!

$\boxed{\text{Exercise 2:}}$ What is (3-D pic) of V_1? V_{-1}?

Another set of representatives of $V(\mathbb{Q})_+/\sim$ is $\{(\frac{x}{z},\frac{y}{z},1)\} \in S^1(\mathbb{Q})$. $(x,y,z) \in$

$\frac{x}{z}, \frac{y}{z} \in \mathbb{Q}$

THAT IS WHY PYTH TRIPLES ARE **FUNDAMENTAL**: Parametrize $\mathbb{Q}$-pts on **circle**. $x^2+y^2=1$. $\in S^1(\mathbb{Q})$.

$\boxed{\text{Exercise 3:}}$ $(x,y) \in S^1(\mathbb{Q}) \Rightarrow s=u$

$\frac{r}{s}$ $\frac{t}{u}$, $r,s,t,u \in \mathbb{Z}$, reduced

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So $\left(\frac{r}{u}, \frac{t}{u}\right) \in S^1(\mathbb{Q}) \Leftrightarrow (r,t,u) \in V(\mathbb{Z})^0_+$

Parametrization of $\underset{(x,y,z)\in}{V(\mathbb{Z})^0_+} \cap \{y \text{ even}\}$:

There exists $(r,s) \in \mathbb{Z}^2_0$ $x=r^2-s^2$, $y=2rs$, $z=r^2+s^2$ (4)

Pf sketch: $r = \sqrt{\dfrac{z+x}{2}}$ $\quad s = \sqrt{\dfrac{z-x}{2}}$

Rmk: holds for $V(\mathbb{R})_+$. Amazing that $(r,s) \in \mathbb{Z}^2$ if $\vec{v} \in V(\mathbb{Z})^\circ_+$.

Why? $y^2 = z^2 - x^2 = \underbrace{(z+x)}_{2a}\underbrace{(z-x)}_{2b}$.

$$y^2 = 4y_1^2 = 2a \cdot 2b.$$

coprime $a \cdot b = y_1^2$.
$\Rightarrow a = r^2,\ b = s^2$.

FLT (for $n=4$):

$$x^4 + y^4 = z^4 \ (\text{in } \mathbb{Z}) \Rightarrow xyz = 0.$$

Fermat (Thm A): $x^4+y^4=z^2$ in $\mathbb{Z} \Rightarrow xyz=0$.

Thm A $\Rightarrow$ FLT because if (x,y,z) solves $x^4+y^4=z^4$ then (x,y,z^2) solves $x^4+y^4=(z^2)^2$.

If $(x^4+y^4-z^2=0) \Rightarrow (x^2,y^2,z) \in V(\mathbb{Z})^{0}_{+}$

$\boxed{\text{Exercise 4:}}$ If $x^4+y^4=z^2$ & any two have a common factor, then all three do.

By Parametrization, $\exists\, r,s \in \mathbb{Z}$ s.t.

$$x^2=r^2-s^2,\quad y^2=2rs,\quad z=r^2+s^2$$

Hwk: Play with these equations & what you know about Pyth triples to see if you can finish the proof.

Hint: $s^2 + x^2 = r^2$ ← new Pyth triple.

has parametrization . . . ?.