Recall: Pythagorean 3 Parametrization!

Lemma: If $(x,y,z)\in\mathbb{Z}^3$ a Pyth triple,
i.e. $x^2+y^2=z^2$ & primitive (no common factors)
Then: $\exists\, r,s\in\mathbb{Z}$, coprime s.t. (assume y even)

$$x=r^2-s^2,\quad y=2rs,\quad z=r^2+s^2$$

[Discovery]: Say we have (x,y,z) with

$$x^2+y^2=z^2 \Rightarrow y^2=z^2-x^2=(z+x)(z-x).$$

$y=6$, $y^2=36=$ 1 · 36, 6 · 6.
2 · 18
3 · 12
4 · 9

If $z+x$ & $z-x$ have a common factor,

$d \mid z+x$ & $d \mid z-x$. $\Rightarrow$ $z+x = d \cdot c$.

$z-x = d \cdot b$

$\Rightarrow d \mid 2z$, & $d \mid 2x$.

$2z = d(b+c)$ | $2x = d(b-c)$

If $\nexists\, \ell \mid x$ & $\ell \mid y$ & $\ell \mid z$. $\Rightarrow$ (x,y,z) is primitive

Pairwise primitive: $(x,y) = \gcd(x,y) = 1 = (y,z) = (x,z)$.

$(2,4,1)$. is primitive, not pairwise primitive.

$(6,10,15)$ " " " " "

But for Pyth triples, primitivity $\Rightarrow$ pairwise primitivity. Reason: if $5 \mid x$ & $5 \mid z$ $\Rightarrow$

$y^2 = z^2 - x^2 \leftarrow 25 \mid z^2 - x^2 \Rightarrow 5 \mid y$. $\square$

<u>Lemma</u>: If (x,y,z) primitive & Pyth triple $\Rightarrow$ pairwise primitive.

<u>Pf</u>: [Contrapositive] Not pairwise primitive $\Rightarrow$ not primitive

$\exists\, \ell \mid x \;\&\; \ell \mid z \Rightarrow \ell^2 \mid z^2 - x^2 = y^2 \Rightarrow \ell \mid y.$ Π.

If $d \mid z+x \;\&\; d \mid z-x \Rightarrow d \mid \underbrace{2z}_{=\,2\cdot p_1^{e_1}\cdots p_n^{e_n}} \;\&\; d \mid \underbrace{2x}_{=\,2\cdot q_1^{f_1}\cdots q_m^{f_m}}$.

But $(x,z)=1 \Rightarrow d = 1$ or 2.

(E.g.: $(x,y,z) = (5,12,13)$, $12^2 = 13^2 - 5^2 = (13+5)(13-5)$

$= \underbrace{18}_{2\cdot 9} \cdot 8.$ $\quad 2\cdot 9\cdot 2\cdot 4.$)

Assumed y was even.

Possible parities of solutions?

$\underline{o^2+o^2=o^2}$ **Lemma:** odd + odd = even

i.e. if a & b are odd, $a+b$ = even.

pf: If a & b are odd, then $a=2k+1$ & $b=2j+1$ with $j,k\in\mathbb{Z}$. Then $a+b=2k+1+2j+1=2(\underbrace{k+j+1}_{\in\mathbb{Z}})$

$\Rightarrow$ $a+b$ is even.

$\boxed{\text{Exercise 3}}$ $odd^2 = odd$ & $even^2 = even$

x	y	z
~~o~~	~~o~~	~~o~~
~~o~~	~~o~~	~~e~~
o	e	o
~~o~~	~~e~~	~~e~~
~~e~~	~~o~~	~~o~~
~~e~~	~~o~~	~~e~~
~~e~~	~~e~~	~~o~~
~~e~~	~~e~~	~~e~~

So no Pyth triple has all 3 odd

If all even, not primitive

If $\Rightarrow$ $\underline{x=2a+1}$, $\underline{y=2b+1, z=2c}$. $\Rightarrow$

& $x^2+y^2 = \underline{4a^2}+\underline{4a}+1+4b^2+\underline{4b}+1 = \underline{4c^2} = z^2$.

(4)

Lemma: If $x^2+y^2=z^2 \Rightarrow$ can't have $x=\text{odd}$, $y=\text{odd}$, & $z=\text{even}$

Pf: $4(a^2+a+b^2+b)+2 = 4\cdot c^2$. impossible.

Assume $y=$ even. $y=2y_1$ & $y^2=z^2-x^2$

$$4y_1^2 = \underbrace{(z+x)}_{2a}\underbrace{(z-x)}_{2b}.$$

$\Rightarrow y_1^2 = ab$ $\gcd(2a,2b)=2$.

$(a,b)=1$.

$$\boxed{k^2\cdot l^2 = (k\cdot l)^2 \ \checkmark.}$$

$p_1^{2e_1}\cdots q_1^{2f_1}\cdots \Rightarrow a$ & b are squares.

$\Rightarrow a=r^2$ & $b=s^2$. $\underbrace{(z+x)+(z-x)}_{2z=} = 2a+2b = 2r^2+2s^2$.

$4y_1^2 = 4ab = 4r^2s^2 \Rightarrow 2y_1 = 2rs = y$.

E.g.: Babylonian tablet row 7:

$d = 3541$, $w = 2291$, $\ell = 2700$. Find r, s

" $z = r^2+s^2$ " $x = r^2 - s^2$ " $y = 2rs$. s.t. $\begin{array}{cc} r & s \\ | & | \\ 54 & 25 \end{array}$

$$y^2 = z^2 - x^2 = \underbrace{(z+x)}\underbrace{(z-x)}$$

$$2a = 5832 \qquad 1250 = 2b$$

$$r^2 = a = 2916 \qquad 625 = b = s^2,\ s = \sqrt{625} = 25$$

$$r = \sqrt{2916} = 54$$

$\boxed{\text{Exercise 4}}$ Compute r, s for row 8.