

Recall: Pythagorean 3-parameterization

Lemma: If $(x, y, z) \in \mathbb{Z}^3$ is a Pyth triple,

i.e. $x^2 + y^2 = z^2$ & primitive (no common factors)

Then: $\exists r, s \in \mathbb{Z}$, coprime s.t. (assume z even)

$$x = r^2 - s^2, \quad y = 2rs, \quad z = r^2 + s^2$$

[Discrete]: Say we have (x, y, z) with

$$x^2 + y^2 = z^2 \Rightarrow y^2 = z^2 - x^2 = (z+x)(z-x).$$

$$y=6, y^2=36 = \begin{matrix} 1. 36 \\ 2. 18 \\ 3. 12 \\ 4. 9 \end{matrix} \quad \text{b.o.}$$

Lemma: If (x, y, z) primitive & Pyth triple

\Rightarrow pairwise primitive.

Pf: [Contra positive] Not pairwise primitive \Rightarrow not primitive

$$\hookrightarrow \exists d \mid x \& d \mid z \Rightarrow d^2 \mid z^2 - x^2 = y^2 \Rightarrow d \mid y. \uparrow$$

If $d \mid z+x$ & $d \mid z-x \Rightarrow d \mid z+z$ & $d \mid z-x$
But $(x, z)=1 \Rightarrow d=1 \text{ or } 2$. $2 \cdot p_1^{e_1} \dots p_n^{e_n} = 2 \cdot q_1^{f_1} \dots q_m^{f_m}$

(E.g.: $(x, y, z) = (5, 12, 13)$, $12^2 = 13^2 - 5^2 = (13+5)(13-5)$

Assumed y was even.

$$= 18 \cdot 8$$

Possible parities of solutions?

$$2 \cdot 9 \cdot 24$$

(3)

$0^2 + 0^2 = 0^2$ Lemma: odd + odd = even

i.e. if a & b are odd, $a+b = \text{even}$.

pf. If a & b are odd, then $a = 2k+1$ & $b = 2j+1$
with $k, j \in \mathbb{Z}$. Then $a+b = 2k+1+2j+1 = 2(k+j+1)$
 $\Rightarrow a+b$ is even.

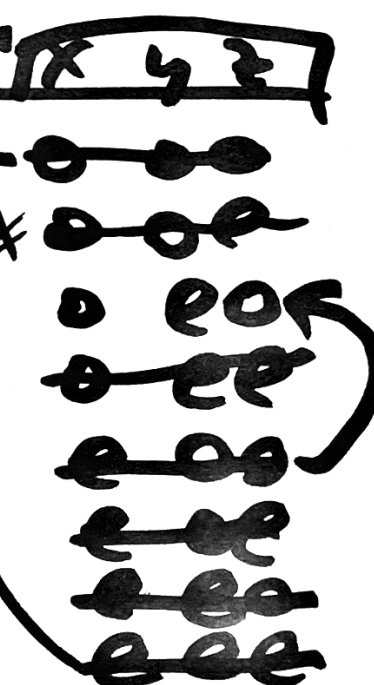
Exercise 3 $\text{odd}^2 = \text{odd}$ & $\text{even}^2 = \text{even}$

So no Pyth triple has all 3 odd

If all even, not primitive

if $x = 2a+1, y = 2b+1, z = 2c. \Rightarrow$

$$x^2 + y^2 = 4a^2 + 4a + 1 + 4b^2 + 4b + 1 = 4c^2 = z^2.$$



Lemma: If $x^2 + y^2 = z^2 \Rightarrow$ can't have $x = \text{odd}$, $y = \text{odd}$, & $z = \text{even}$.

Pf: $4(a^2 + b^2) + 2 = 4c^2$. impossible.

Assume $y = \text{even}$. $y = 2y_1$, & $y^2 = z^2 - x^2$

$$\Rightarrow y^2 = ab \quad \gcd(2a, 2b) = 2. \quad 4y_1^2 = \underbrace{(z+x)}_{2a} \underbrace{(z-x)}_{2b}$$

\downarrow
 $p_1^{2e_1} \dots q_1^{2f_1} \dots \Rightarrow a \& b$ are squares.

$$\boxed{k^2 \cdot l^2 = (k \cdot l)^2 \cdot v}$$

$$\Rightarrow a = r^2 \text{ \& } b = s^2. \quad \underbrace{(z+x) + (z-x)}_{2z} = 2a + 2b = 2r^2 + 2s^2$$

$$4y_1^2 = 4ab = 4r^2s^2 \Rightarrow 2y_1 = 2rs = y.$$

E.g.: Babylonian tablet row 7:

$$d = 3541, \quad u = 2291, \quad l = 2700.$$

$$z = r^2 + s^2, \quad x = r^2 - s^2, \quad y = 2rs.$$

$$\text{Find } r, s \\ \text{s.t. } \begin{matrix} 1 & 1 \\ 54 & 25 \end{matrix}$$

$$y^2 = z^2 - x^2 = \underbrace{(z+x)}_{2a} \underbrace{(z-x)}_{1250=2b}$$

$$2a = 5832 \quad 1250 = 2b$$

$$r^2 = a = 2916$$

$$625 = b = 5^2, \quad 5 = \sqrt{625} = 25$$

$$r = \sqrt{2916} = 54$$

Exercise 4

Compute r, s for row 8.