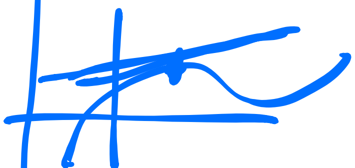
$f : \mathbb{C} \to \mathbb{C}$.

When is f diff'ble?

$f : \mathbb{R}^2 \to \mathbb{R}^2$

Calculus $\Leftrightarrow$ linearization



f is diff'ble at z if

$$\underbrace{f(z+h)} = \underline{f(z)} + \underline{L \cdot h} + \underbrace{o(|h|)}_{E}.$$

Here $E = o(|h|) \Leftrightarrow \frac{E}{|h|} \to 0$, as $\underbrace{|h| \to 0}$.

$f : \mathbb{R}^n \to \mathbb{R}^m$ is diff'ble if locally affine-linear

$$f(\underbrace{x_1, \dots, x_n}) = (f_1(\vec{x}), \dots, f_m(\vec{x})).$$

$$f(\vec{z} + \vec{h}) = f(\vec{z}) + \underline{L} \cdot \underline{h} + o(|h|)$$

where $L = \left(\frac{\partial f_i}{\partial x_j} \right)_{ij}$

$$f(\underbrace{x, y}_{x+iy}) = \underset{\text{Re } f}{u} + i \underset{\text{Im } f}{v},$$

$$\boxed{L = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}} = (u, v).$$

$\hookrightarrow \boxed{\dfrac{f(z+h)-f(z)}{h}} = L + o(1) \to L$

In $\mathbb{R}^4$, this makes no sense, but we can do it w/ $\mathbb{C}$!

Def: f is <u>holomorphic</u> at z if

$$\lim_{h \to 0} \frac{f(z+h)-f(z)}{h} \text{ exists } "f'(z)"$$

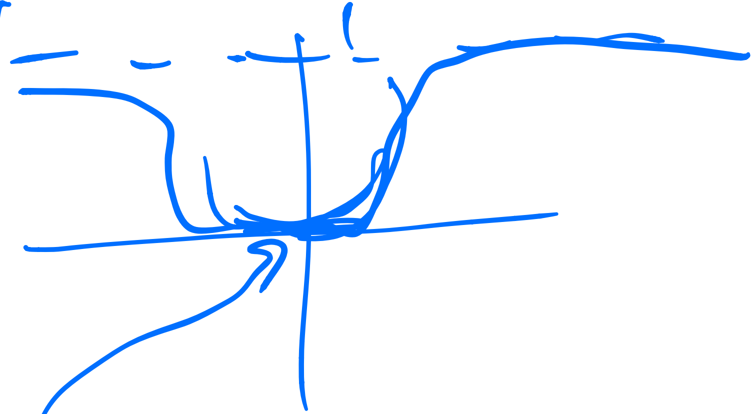
Also f is continuous at <u>z if: $\forall \varepsilon > 0$</u>

$\exists \delta > 0 \; \forall |h| < \delta$

$|f(z+h) - f(z)| < \varepsilon$,

to graph f, need $\mathbb{R}^4$.

To see f show "before" & "after"



As $z \to 0$, $-1/z^2 \to -\infty$

As $|z| \to \infty$, $f \to e^0 = 1$.

E.g.: $f(z) = \begin{cases} e^{-1/z^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ over $\mathbb{R}$?

f is cont.

Is f diff'ble at 0?

$$f' = \underbrace{e^{-1/z^2}}_{\to 0} \underbrace{(-1)(-2)z^{-3}}_{\to \infty}$$

As $z \to 0$?

$f' \to 0$ as $z \to 0$.

$$f^{(n)} = \underline{e^{-1/z^2}}\, \underline{P_n(z^{-1})}.$$

$\to 0$ as $z \to 0$.

$f \in C^\infty(\mathbb{R})$.

"smooth".

$$g = \begin{cases} e^{-1/z^2}, & z > 0 \\ 0, & z \leq 0 \end{cases}$$

"smooth bump function".

[partition of 1].

g is smooth &

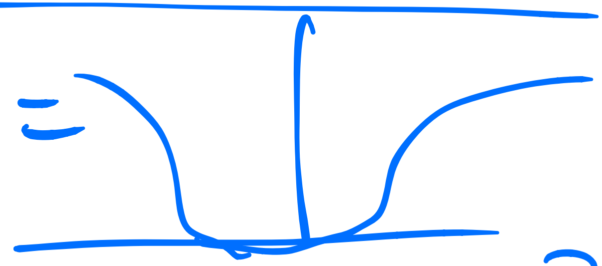
$f \equiv g$ on $z \geq 0$.

$f \neq g$ on $z < 0$.

Both f & g extend $e^{-1/z^2}\big|_{\mathbb{R}_+}$ to all $\mathbb{R}$.

But $f \neq g$.

CAN'T happen for holomorphic functions

$f =$ 

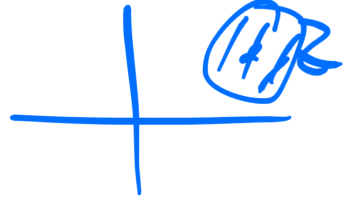
Is f analytic?

Power series rep's f @ 0.

Def: f is analytic at z if $\boxed{\exists\, \varepsilon>0}$ $\exists\, a_0, a_1, a_2, \ldots$ s.t. $\forall\, |h|<\varepsilon$,

$$a_0 + a_1(h) + a_2(h)^2 + \cdots = \sum_{i=0}^{\infty} a_i (h)^i \text{ conv. absolutely}$$

& $= f(z+h)$.



Is $f = \begin{cases} e^{-1/z^2} & \\ 0 & z=0 \end{cases}$ analytic at 0?

All Taylor coeffs are zero!

$a_0 = a_1 = a_2 = \cdots = 0$.

$\nexists\, \varepsilon>0$ on which

Taylor series describes f. Not analytic

Fact: holo $\Rightarrow$ analytic

what about $f(z) = e^{-1/z^2}$

for $z \in \mathbb{C}$?

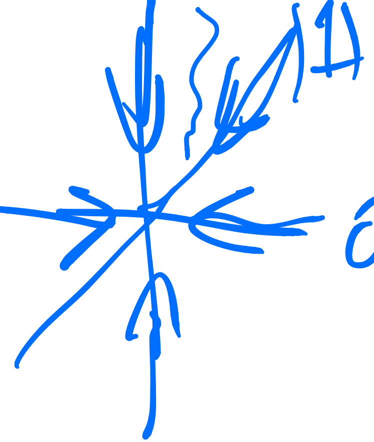
As $z \to 0$, ~~$f \to 0$~~.

As a $\mathbb{C}$-valued function, NOT EVEN continuous!

$z \to 0$ from $\mathbb{R}$, $f \to 0$.

Let $z = iy$, $y \to 0$.

$$e^{-1/(iy)^2} = e^{1/y^2} \to \infty$$



$z = e^{\pi i/4}\cdot\lambda.$

$e^{-1/(e^{\pi i/4}\lambda)^2} = \boxed{e^{-1/i\lambda^2}}$

$|\cdot| = |e^{i\cdot\frac{1}{\lambda^2}}| = 1$

"essential singularity"

$f(x,y) = (u(x,y), v(x,y))$

Again f hol $\Rightarrow$

$$\frac{f(z+h)-f(z)}{h} \xrightarrow[h\to 0 \text{ in } \mathbb{C}]{} f'(z)$$

$f(z+h) = f(z) + L\cdot h + o(h^{1+\epsilon})$

$L = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ $f = u + iv$

What happens if

$h = h_1 + ih_2$ $h = h_1^{\in\mathbb{R}} \to 0.$

$$\frac{f(z+h_1)-f(z)}{h_1} \to \frac{\partial f}{\partial x}$$

$$\shortparallel$$

$$f'(z) = u_x + i v_x.$$

If $h = i h_2$, $h_2 \in \mathbb{R}$, $h_2 \to 0$

$$\frac{f(z+ih_2)-f(z)}{ih_2} \to \frac{1}{i}(u_y + i v_y)$$

$$\shortparallel$$

$$f'(z).$$

Thm (Cauchy-Riemann equations):

If $f = u + iv$ is hol at $z = x+iy$, then: $\boxed{u_x = v_y \ \& \ v_x = -u_y.}$

$$\Rightarrow L = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

What does this mean geometrically?

QR decomposition?

Any $L = \underbrace{Q}_{\text{orthogonal}} \cdot \underbrace{R}_{\text{upper } \Delta}$

$$\begin{pmatrix} x & x \\ 0 & x \end{pmatrix}$$

For 2x2

$$L = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{K} \underbrace{\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}}_{N}$$

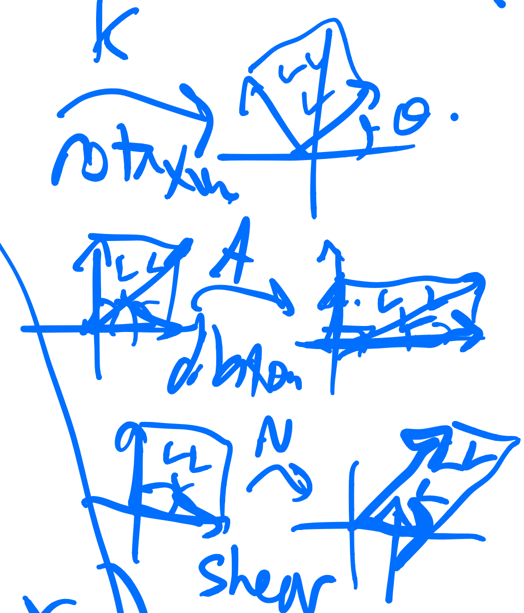
$Q \cdot Q^T = I$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+b^2 & x \\ x & x \end{pmatrix}$

K = "compact".

A = "abelian" = "diagonal"

N = "unipotent".

= eigenvalues are 1.



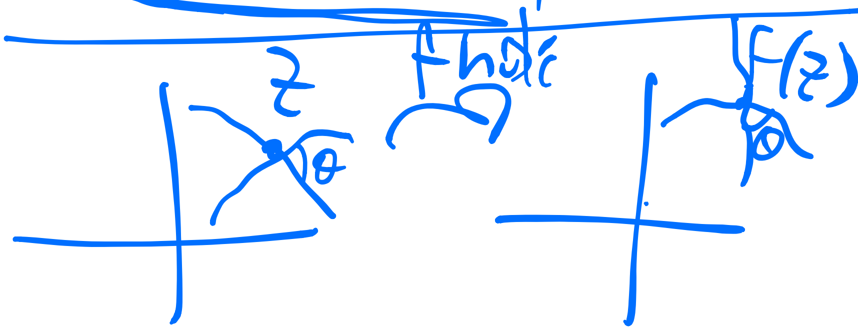
Not Δ.

Exercise: $L = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$.

$\Rightarrow \xi = 0$ No shearing & uniform dilation.

$\Rightarrow$ locally, angles are preserved

"conformal"

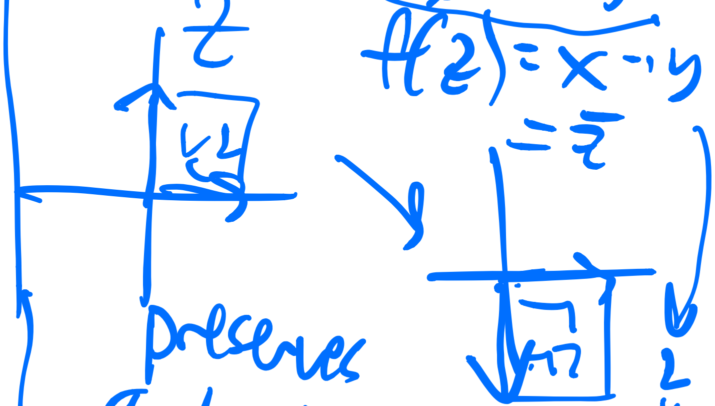


$\Rightarrow L = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix}$.

Jacobian = $\boxed{\det L}$.

$= A^2 + B^2 = u_x^2 + u_y^2 > 0$.

$f(z) = x - iy = \bar{z}$



preserves angles but not orientation

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Prove: f hol'c ⇒

orientation preserved

$$\det L = u_x^2 + u_y^2 = u_x^2 + (-v_x)^2$$

$$= u_x^2 + v_x^2 = |u_x + iv_x|^2.$$

$$\frac{\partial}{\partial x} f = f'(z).$$

$$\det L = |f'(z)|^2.$$

$$\frac{\partial}{\partial x} f = u_x + iv_x.$$

$$f(x,y) = (u(z), v(z)).$$

Let $f_1(z, \bar{z})$ "$\bar{z}$"

$$= f\Big(\underbrace{\frac{z+\bar{z}}{2}}_{x}, \underbrace{\frac{z-\bar{z}}{2i}}_{y}\Big)$$

What should $\frac{\partial}{\partial z}$ mean?

$$\frac{\partial}{\partial z} f_1(z, z_1) = \frac{\partial}{\partial z} f\left(\frac{z+\bar z}{2}, \frac{z-\bar z}{2i}\right)$$ free

$$= f_x \cdot \frac{1}{2} + f_y \cdot \frac{1}{2i}.$$

As differential operator

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right).$$

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

formal symbol

$$\boxed{\frac{\partial}{\partial \bar z}}(u+iv) = \frac{1}{2}\left((u_x+iv_x) - \frac{1}{i}(u_y+iv_y)\right)$$

If f is hol'c

$$\Rightarrow u_x = v_y \ \& \ u_y = -v_x$$

$$\Rightarrow \frac{\partial}{\partial \bar z} f = \frac{1}{2}\left[u_x + i v_x - \frac{1}{i}[u_y + i v_y]\right]$$

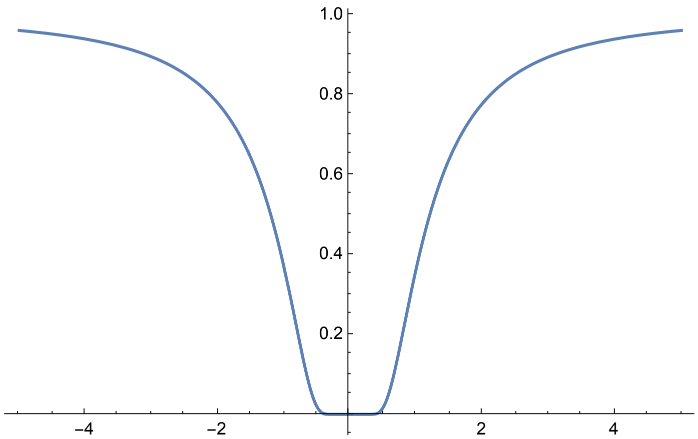
$$= \frac{1}{2}\left[\underbrace{(u_x - v_y)}_{0} + i\underbrace{(v_x + u_y)}_{0}\right]$$

$$= 0.$$ C-R.

f hol'c $\Rightarrow$ "only a function of z, $\boxed{\frac{\partial}{\partial \bar z} = 0}$".

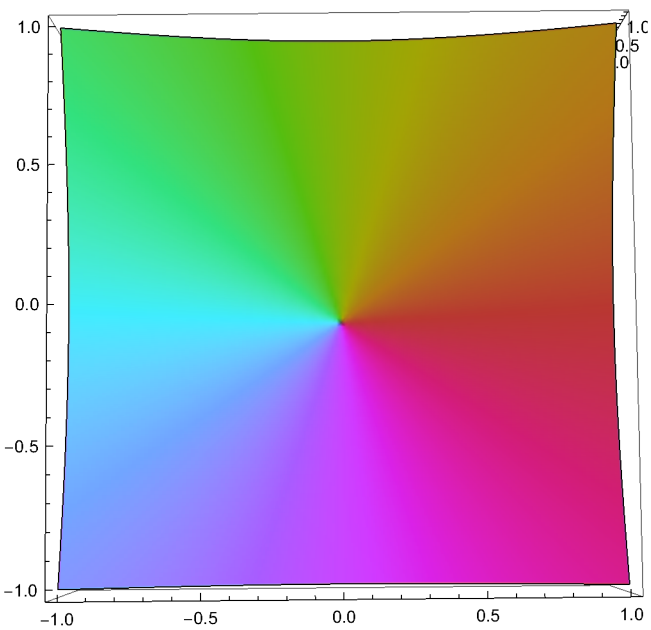
In[∘]:= `Plot[E ^ (-1 / z ^ 2), {z, -5, 5}]`

Out[∘]=



In[∘]:= `ComplexPlot3D[z, {z, -1 - I, 1 + I}]`

Out[∘]=

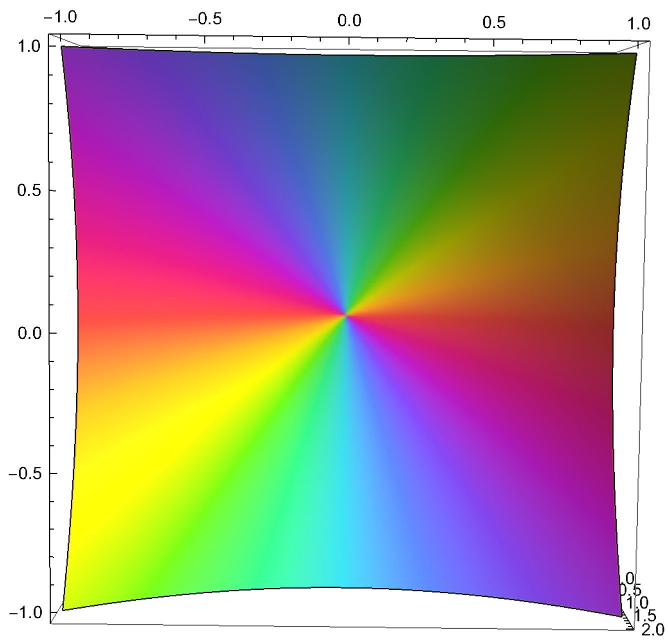


z = rho E ^ (i theta), rho = | Z |, theta = "argument"

z ^ 2 = rho ^ 2 E ^ (i 2 theta)

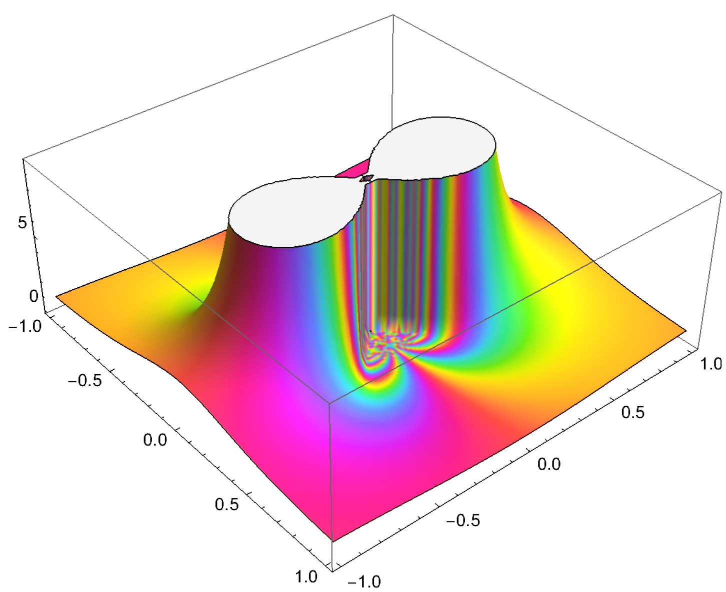
In[∘]:= `ComplexPlot3D[z^2, {z, -1 - I, 1 + I}]`

Out[∘]=



In[1]:= `ComplexPlot3D[E^(-1 / z^2), {z, -1 - I, 1 + I}]`

Out[1]=



In[∘]:=
```
Show[ComplexPlot3D[E^(-1/z^2), {z, -1 - I, 1 + I}],
 ComplexPlot3D[1, {z, -1 - I, 1 + I}]]
```

Out[∘]=