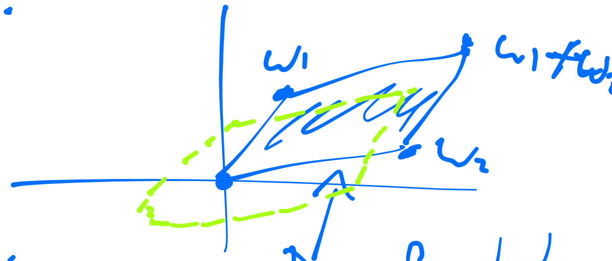
Last time: "elliptic" functions $\Leftrightarrow f : \mathbb{C}/\Lambda \to \mathbb{C}$ merć

doubly periodic.

$\Lambda = \langle \omega_1, \omega_2 \rangle$, $\omega_2/\omega_1 \notin \mathbb{R}$



D = fund dom $= D(\Lambda)$

- f elliptic & hol $\mathbb{C}$ $\Rightarrow$ const.
  order $=0$.

- E.g.: $f(z) = \sum_{\lambda \in \Lambda} g(z+\lambda)$ converges if, e.g.,

  non const $\Rightarrow$ essential at ∞

  $g = \frac{P}{Q}$, $\deg Q \geq \deg P + 3$.

  $= \frac{1}{z^3}$, $= \frac{z^2 - 17}{z^6 + 12z^4 + 1}$.

  is merć & elliptic
  $\to$ order ≥ 3.

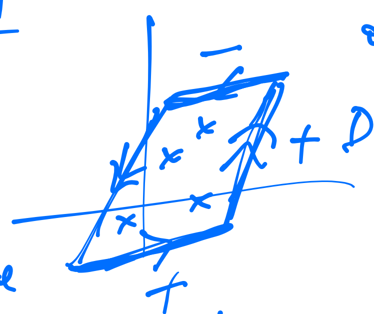
Def: **Order** of elliptic f on Λ is # poles (w/ mult) in D.

~~order $=1$?~~ order $=2$?

Thm: DNE f elliptic & order $=1$.

pf: If f elliptic (wrt Λ) & merć, look at (assume no poles on ∂D)

$$0 = \frac{1}{2\pi i} \int_{\partial D} f(z)\,dz = \sum_{\alpha_i \text{ poles in } D} \operatorname{Res}_{\alpha_i} f = \operatorname{Res}_{\alpha} f$$



$$\frac{1}{z^2} + \frac{\boxed{0}}{z} + 0 + 0z + \cdots$$

$\Rightarrow$ can't have single simple pole!.

Q: Does there exist elliptic f of order $=2$?

Guess: $\sum_{\lambda \in \Lambda} \frac{1}{(z+\lambda)^2}$? . Does not converge abs.

$\lambda \in \Lambda$

Back to 1-d: $\overset{\pi \cot \pi z \text{ ''}}{=} \displaystyle\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)}$ " harmonic series diverges.

Does not converge absolutely! Idea 1: interpret infinite sum as limit of symmetric sum

$$= \lim_{N\to\infty} \underbrace{\sum_{|n|\le N} \frac{1}{z+n}}_{} \to \frac{1}{z} + \underbrace{\sum_{1\le n\le N} \left(\frac{1}{z+n} + \frac{1}{z-n}\right)}_{\frac{2z}{z^2-n^2}}$$

$$= \frac{1}{z} + \sum_{1\le n<\infty} \frac{2z}{z^2-n^2}. \leftarrow \text{converges abs.}$$

Idea 2: $\displaystyle\frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \underbrace{\left[\frac{1}{z+n} - \frac{1}{n}\right]}_{}$ $\swarrow$ Can sum in any order

If interpreted symmetrically Idea 2 = Idea 1. $\quad \frac{-z}{n(z+n)} = O_z\left(\frac{1}{n^2}\right)$ as $n\to\infty$.

1870s: Weierstrass: $\wp(z) = \wp_\Lambda(z) = \displaystyle\sum_{\lambda\in\Lambda} \frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2}$ $\not=$

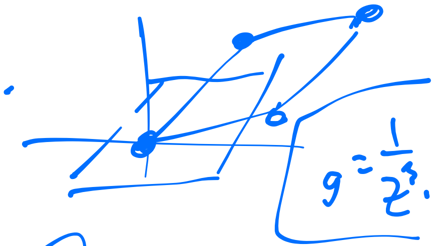
$$\boxed{\frac{1}{z^2} + \sum_{\lambda\in\Lambda^*} \underbrace{\left[\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2}\right]}_{}} \leftarrow \text{series converges abs (\& unif on cpta).}$$

$$\frac{-z^2-2z\lambda}{\text{same}} = \frac{\lambda^2-(z+\lambda)^2}{\lambda^2(z+\lambda)^2} \leftarrow \approx \frac{|\lambda|}{|\lambda|^4} = \frac{1}{|\lambda|^3}, \quad \Downarrow \ \wp \text{ is}$$

meri'c.

No longer obvious: $\wp(z+\lambda_0) = \wp(z)$?

Thm: $\wp$ is elliptic of order 2.



Note: $\wp$ is even $\wp(z) = \wp(-z)$.

pf: $\wp'(z) = \frac{-2}{z^3} + \sum_{\lambda \in \Lambda^*} \left[\frac{-2}{(z+\lambda)^3} + 0 \right] = -2 \sum_{\lambda \in \Lambda} g(z+\lambda)$

$\Rightarrow$ elliptic of order 3. $\Rightarrow$ $\wp'(-z) = -\wp'(z)$ (odd).

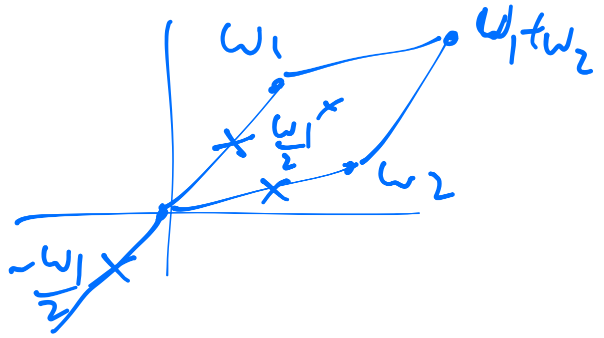
Consider $F(z) = \wp(z)$, $G(z) = \wp(z+\lambda_0)$.

$F' = \wp' = G'$. $\quad G'(z) = \wp'(z+\lambda_0) = \wp'(z)$.

So $F - G$ has deriv 0. & hol'c.

$\Rightarrow$ $\wp(z) = \wp(z+\lambda_0) + c$.

for $\lambda_0 = \omega_1$, for $z = -\frac{\omega_1}{2}$: $\quad \wp\left(-\frac{\omega_1}{2}\right) = \wp\left(\frac{\omega_1}{2}\right) + c$



But $\wp$ is even! So $c = 0$.

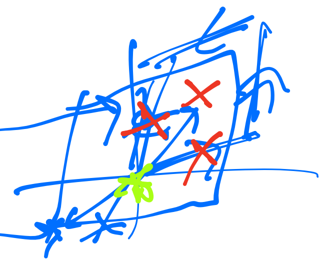
$\Rightarrow$ $\wp(z+\omega_1) = \wp(z)$, same $\omega_1 \to \omega_2$. $\Rightarrow \Lambda$.

Thm: If f is elliptic of order n, then f has n poles in D & n zeros in D?

Pf: $0 = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)}\, dz = \#\text{zeros} - \#\text{poles}$.

wiggle D $\longrightarrow$ ∂D

So f has no zeros/poles on ∂D



$\wp'$ is odd, so $\wp'\left(-\frac{\omega_1}{2}\right) = -\wp'\left(\frac{\omega_1}{2}\right) = 0$.

$= \wp'\left(\frac{\omega_1}{2}\right)$

$\wp'$ vanishes at: $\underbrace{\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}}$ (simple zeros)

$\wp'$ has simple poles at: $\underbrace{0, 0, 0}$,

Let $\wp\left(\frac{\omega_1}{2}\right) = e_1$, $\wp\left(\frac{\omega_2}{2}\right) = e_2$, $\wp\left(\frac{\omega_1+\omega_2}{2}\right) = e_3$

Claim: function $\underbrace{\wp(z) - e_1}$ has a double root at $z = \frac{\omega_1}{2}$.

elliptic of order 2

$= 0 + \cancel{\wp'\left(\frac{\omega_1}{2}\right)}^{\,0}\left(z - \frac{\omega_1}{2}\right) + (\underline{\qquad\qquad})$

Why not more than double root? $\neq 0$.

In fact, no other point in D except $\frac{\omega_1}{2}$ has $\wp(z) = e_1$.

$\Rightarrow e_1 \neq e_2 \neq e_3$

Look at: $F(z) = \big(\underbrace{\wp(z) - e_1}_{\sim 1/z^2}\big)\big(\underbrace{\wp(z) - e_2}_{\sim 1/z^2}\big)\big(\underbrace{\wp(z) - e_3}_{\sim 1/z^2}\big)$.

F is elliptic of order 6 at 0, double zeros at $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$.

Near 0, $F(z) = \frac{1}{z^6} + \dots$

Alt: $G(z) = \underbrace{\wp'(z)}_{-\frac{2}{z^3}}{}^2 \simeq \frac{4}{z^6} + \dots$

order 6 pole at 0, double zeros at $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$.

So F/G is elliptic of order 0 $\Rightarrow F/G = C = 1/4$.

Thm: $\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$

e_i distinct.

elliptic curve $\longrightarrow$ $y^2 = 4(x - e_1)(x - e_2)(x - e_3)$.

$\boxed{x = \wp_\Lambda(z), \quad y = \wp'_\Lambda(z)}$

parametrized by $(\wp, \wp')$.

$\boxed{x = R\cos t \\ y = \sin t}$ $\boxed{\frac{x^2}{R^2} + y^2 = 1}$

Moreover: Claim!: Every elliptic function is a rational function of $\wp$, $\wp'$.

$f(z) = R(\wp(z), \wp'(z))$ rational funct $\frac{U}{V}$

Claim 2: Every even elliptic function is $R(\wp)$.

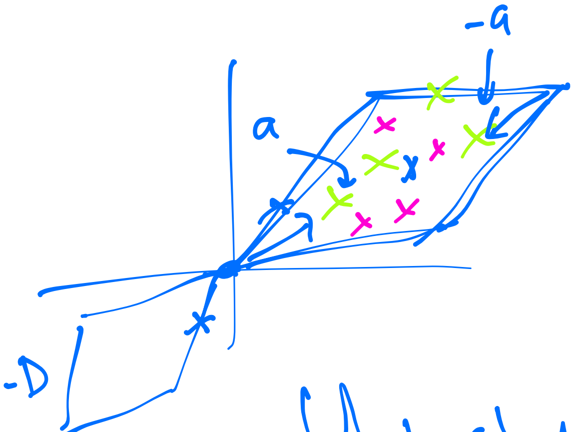
pf: Given f even & elliptic (on Λ).

So at $z \equiv 0$, f { regular / zero ← even order / pole ← }

So $f \cdot \wp^m(z)$ is regular $\neq 0$ at 0. $\left(m = -\frac{\text{ord}}{2}\right)$

$\wp$ has order 2 pole at 0.

Can assume $f \neq 0$ at 0, no pole there.



since f is even, if f has zero at $a \in D$, must also have zero at $-a + \omega_1 + \omega_2 \in D$.

What about $\wp(z) - \wp(a)$ also has zeros at $z = a, -a$.

Say f has zeros at $\pm a_1, \dots, \pm a_m$ (order $f = 2m$)

poles at $\pm b_1, \dots, \pm b_m$.

Let $G(z) = \dfrac{\prod_{j=1}^m (\wp(z) - \wp(a_j))}{\prod_{j=1}^m (\wp(z) - \wp(b_j))}$ has same zeros & poles as f!

So f/G is hol $\Rightarrow$ const. So $f = R(\wp)$.

If f ell'ptic & not even/odd,

$$f = f_e + f_o, \qquad f_e(z) = \frac{f(z)+f(-z)}{2}$$

$$R(p) \qquad f_o(z) = \frac{f(z)-f(-z)}{2}.$$

So $f_o/p' = $ even, $= R_1(p)$. $\Rightarrow f$ is $R(p, p')$