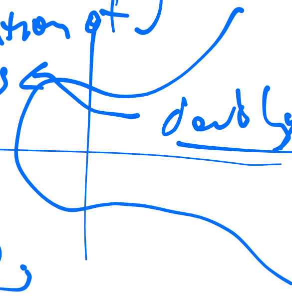
"elliptic curves" ← "elliptic integrals"

"elliptic functions".

$$\boxed{y^2 = x^3 + Ax + B.}$$

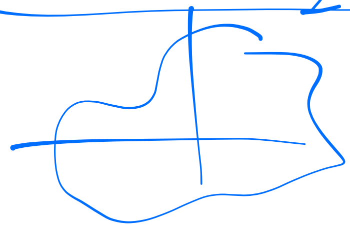
Want: parametrization of "hidden" symmetries

doubly periodic



quadratic curves:

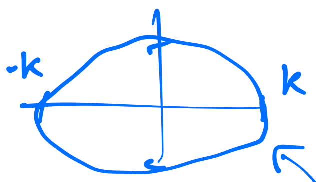
$$Ax^2 + Bxy + Cx + Dy^2 + Ey + F = 0,$$



All such curves are conic sections

Apollonius (250 BCE).

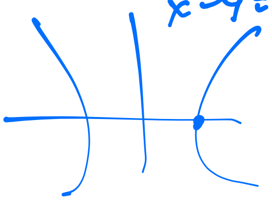
- ellipse $\boxed{\frac{x^2}{k^2} + y^2 = 1}$
- hyperbola $\boxed{x^2 - y^2 = 1}$, $xy = 1$
- parabola $y = x^2$. , $x = x$, $y = x^2$.



parametrize $x = k\cos t$, $y = \sin t$ ← transcendental

$\underline{\sin}(z + 2\pi)$ function on $\mathbb{C}/2\pi\mathbb{Z}$

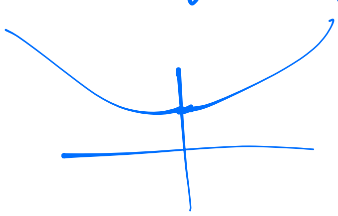
$x^2 - y^2 = 1$

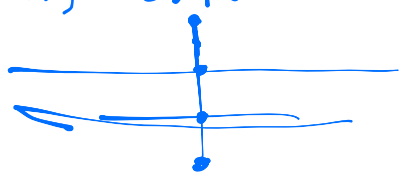


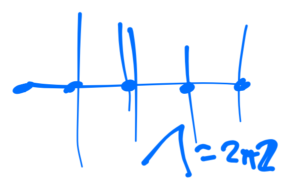
$x = \cosh t$, $y = \sinh t$ ← over $\mathbb{C}/2\pi i\mathbb{Z}$

$$\cosh t = \frac{e^t + e^{-t}}{2}.$$

$\cosh(t + 2\pi i) = \cosh t.$



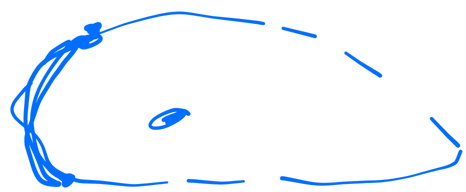




$\Lambda = 2\pi\mathbb{Z}$

1650s Newton gravity inverse sq.

⇒ Kepler that

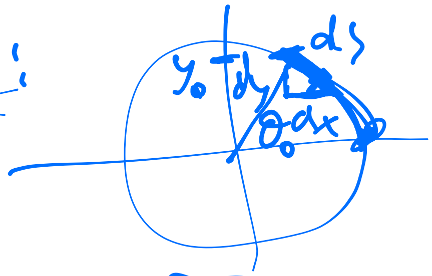


elliptical orbits.

How far does Earth travel over time?

i.e. what is arclength of ellipse?

Simpler: circle:




$$\frac{x^2}{k^2}+y^2=1$$

$$x=k\sqrt{1-y^2}.$$

$l=\int ds$, $\quad ds^2 = dx^2+dy^2\frac{1-y^2}{1-y^2}$

$$= \frac{dy^2}{1-y^2}\left(1+(k^2-1)y^2\right).$$

$$dx = \frac{k}{2}(1-y^2)^{-1/2}(-2y)\,dy$$

$$dx^2 = \frac{k^2y^2dy^2}{1-y^2}.$$

$$l = \boxed{\int_0^{y_0}\frac{dy\sqrt{1+(k^2-1)y^2}}{\sqrt{1-y^2}}} = \arcsin y_0.$$

$$= \theta_0$$

"elliptic integral"

$\boxed{\sin\theta_0} = y_0$

"elliptic function".

Polar coords?

$x = r\cos\theta$, $\quad dx = dr\cos\theta - \sin\theta\, r\,d\theta$

$y = r\sin\theta$, $\quad dy = dr\sin\theta + \cos\theta\, r\,d\theta$.

$$dx^2+dy^2 = dr^2\cos^2\theta + r^2\sin^2\theta\,d\theta^2 - \cancel{2r\cos\theta\sin\theta\,dr\,d\theta}$$
$$dr^2\sin^2\theta + r^2\cos^2\theta\,d\theta^2 + \cancel{2r\sin\theta\cos\theta\,dr\,d\theta}$$

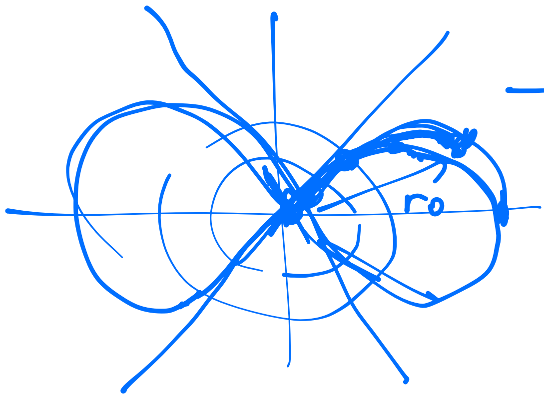
$$ds^2 = dr^2 + r^2d\theta^2 = d\theta^2$$

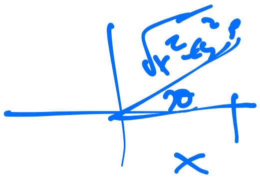
$r=1$,

$dr=0$,

$$\int ds = \int_0^{\theta_0} d\theta = \theta_0.$$

One more example: ~~lemnis~~ Lemniscate $\Rightarrow$ $\boxed{r^2 = \cos(2\theta) = \cos^2\theta - \sin^2\theta}$






Convert to $x, y$.

$r^2 = x^2 + y^2$

$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$

$$\boxed{(x^2-y^2) = (x^2+y^2)^2}$$

$$\cos^2\theta - \sin^2\theta = \frac{x^2 - y^2}{x^2+y^2} = x^2 + y^2,$$

$$ds^2 = \frac{r^4}{1-r^4}dr^2 + r^2 d\theta^2 = \frac{dr^2}{1-r^4}$$

$$\cancel{2} r\,dr = -\sin(2\theta)\cancel{2}d\theta$$

$$\ell = \int ds = \int_0^{r_0} \frac{dr}{\sqrt{1-r^4}}$$

$$\frac{r^2 dr^2}{1-r^4} = \underbrace{\sin^2(2\theta)}_{(1-r^4)} d\theta^2$$

"elliptic integral".

Legendre: 1780

lemniscate $\rightarrow$ 1st kind $F(u) = \int_0^u \frac{dx}{\sqrt{(1-x^2)(1+k^2x^2)}}$

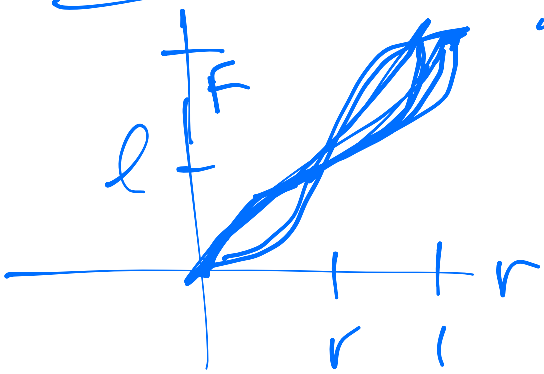
eg ellipse $\rightarrow$ 2nd kind $E(u) = \int_0^u \frac{1-k^2x^2}{\sqrt{(1-x^2)(1-k^2x^2)}}dx$

3rd kind $\Pi(u) = \int_0^u \frac{1}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}dx.$

Abel / Jacobi (Gauss): ~~elliptic integrals~~

elliptic function = inverse of $\int$.

Eg: $F : r \mapsto \int_0^r \frac{dx}{\sqrt{1-x^4}}$, $F'(r) = \frac{1}{\sqrt{1-r^4}} > 0$.




$F^{-1} = sn$ "sine amplitude" function.

$\sqrt{1-sn^2} = cn$

dn

$\ell = F(r)$

$sn(\ell) = r$, $sn(F(r)) = r$ $\quad \frac{d}{dr}$

$$sn'(F(r)) \cdot F'(r) = 1$$

$$sn'(\ell) = \frac{1}{F'(sn(\ell))} = \sqrt{1-(sn(\ell))^4}$$

$(sn')^2 = 1 - sn^4$. $\xrightarrow{\text{Jacobi}}$ $\boxed{\text{double periodicity}}$

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More generally:

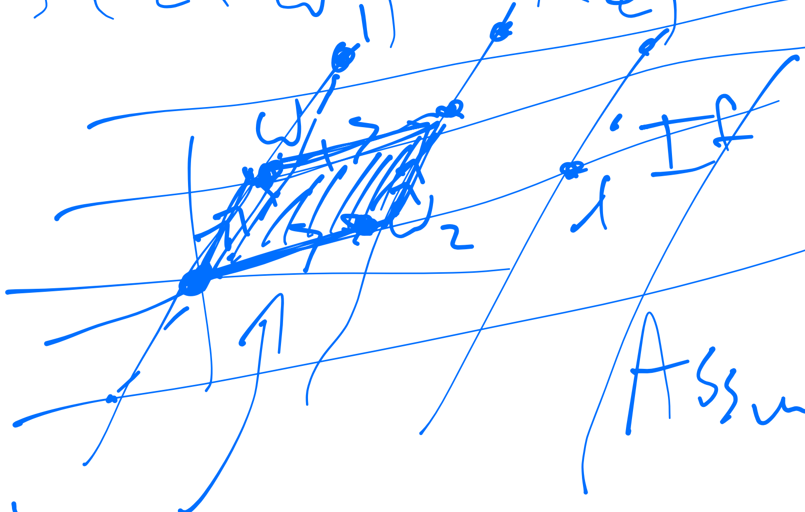
elliptic integral $\longrightarrow$ $\boxed{\int R(t, \sqrt{P(t)})\, dt}$

$\int \frac{1-k^2t^2}{\sqrt{1-t^2}}$ rational funct deg $\leq 4$ poly.

elliptic functions — doubly periodic hol/merc function on $\mathbb{C}$,

$f$ is doubly periodic if $\exists \omega_1, \omega_2$ $\forall z$,

$$f(z+\omega_1) = f(z) = f(z+\omega_2).$$



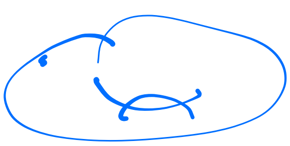

If $\omega_1/\omega_2 \in \mathbb{R}$, not real double periodicity.

Assume $\omega_1/\omega_2 \notin \mathbb{R}$.

"fundamental domain" tiles $\mathbb{C}$,

So $f : \underbrace{\mathbb{C}/\Lambda}_{\text{torus}} \to \mathbb{C}$, $\Lambda = \langle \omega_1, \omega_2 \rangle$ lattice in $\mathbb{C}$

$= \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$.



Does any such non const function exist?

$f$ ∄ hol'c?. Would be entire & bdd $\Rightarrow$ const.

Try "averaging" over $\Lambda$.

$$f(z) := \sum_{\lambda\in\Lambda} g(z+\lambda)$$

as long as this converges absolutely,

$f(z+\lambda_0) = f(z)$, $\quad \lambda = \omega_1 n + \omega_2 m$.

Suppose $|g(z)| \le \dfrac{C}{1+|z|^{\alpha}}$ $\quad (\alpha > 2)$

What conditions on $\alpha$ guarantee abs conv.


$\#\{\lambda\in\Lambda, R\le|\lambda|<R+1\} \ll R$.

$$\sum_{\lambda\in\Lambda}\left|g\left(\overset{0}{\overset{\downarrow}{z}}+\lambda\right)\right|$$

$$\le 1+\sum_{R\ge 1}\ \sum_{R\le|\lambda|<R+1}|g(\lambda)|$$

$$\le 1 + \sum_{R \ge 1} \left( \sum_{R \le |\lambda| < R+1} \frac{1}{|R|^{\alpha}} \right) R,$$

Converges when $\alpha - 1 > 1$

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$g(z) = \dfrac{1}{1+z^3}$ any rational $\dfrac{P(z)}{Q(z)}$

with $\deg P \le \deg Q + 3$

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easy get cubic decay at $\infty$.

Can we get quadratic decay at $\infty$?