Recall: "Riemann Zeta Function"

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \qquad (\mathrm{Re}(s) > 1)$$

$$\left(\begin{array}{c}\text{Euler}\\ \text{Product}\end{array}\right) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right)$$

Euler 1737: $\sum_p \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \infty$.

($\Rightarrow \exists$ ∞ly many primes).

Dirichlet 1837: $(a,q)=1 \Leftrightarrow \sum_{p \equiv a(q)} \frac{1}{p} = \infty$ ($\Rightarrow$ ∞ly many).

Gauss 1790/1800: Extensive tables of primes & logs.

Gauss Conjecture: $\pi(x) := \#\{p < x\} \sim \cancel{\frac{x}{\log x}}$, i.e. $\frac{\pi(x)\log x}{x} \to 1$

Riemann 1859: Strategy for proving $\longrightarrow \sim \int_2^x \frac{dt}{\log t} = Li(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$

"logarithmic integral function"

1896 Hadamard, de la Vallée Poussin: "Prime Number Theorem"

Key observations/discoveries: • ζ has meromorphic continuation, hol'c except for a pole at $s=1$. (where harmonic series diverges) • Key role played by zeros of ζ.

Lemma: $\zeta(s) \neq 0$ on $\mathrm{Re}(s) > 1$.

pf: $\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \leftarrow$ convergent vanishes $\Leftrightarrow$ some term does. $\frac{1}{1 - \frac{1}{p^s}} \neq 0$. $\leftarrow$ vanishes

$\prod_n F_n(s)$, $|F_n(s)-1| < c_n$, $\sum c_n < \infty$.

$\Rightarrow$ converges & $=0 \Leftrightarrow F_n(s)=0$.

If $1=\frac{1}{p^s}$, $p^s=1$, $e^{s\log p}=1$.

$\Leftrightarrow s\log p = n\cdot 2\pi i$

$\sigma = Re(s) \Leftarrow s = \frac{n\cdot 2\pi i}{\log p}$

<u>Recall:</u> Suppose $\{F_n\}: \Omega \to \mathbb{C}$, & $\exists \{c_n\}$ with $\sum c_n < \infty$ s.t. $\forall s \in \Omega$, $|F_n(s)-1| < c_n$.

Then $\prod_n F_n(s)$ converges to holo function & $=0 \Leftrightarrow$ some term is

To get this analytic continuation, need to study "Mellin transform" (cousin of Fourier)

<u>Power Series:</u> $\sum a_n z^n$.

<u>"Dirichlet series":</u> $\sum_{n\geq 1} \frac{a_n}{n^s}$. $(a_n = 1 \Rightarrow \zeta(s))$.

$n^s \leftarrow e^{s\cdot\log n}$

Standard FT: $(\mathbb{R}, +)$

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) \underbrace{e^{-2\pi i x\cdot\xi}}_{\text{character}} \underbrace{dx}_{\text{"Haar measure"}}.$$

invariant $x \mapsto x+7$: $dx \mapsto dx$

$\boxed{s = \sigma + it, \ t \in \mathbb{R}}$

Def: Mellin transform: $\tilde{f}(s) = \int_0^\infty f(t)\, t^s \cdot \frac{dt}{t}$

$(\mathbb{R}_{>0}, \cdot)$

"character" $(t_1 t_2)^s = t_1^s \cdot t_2^s$

"invariant measure": $t \mapsto \lambda t$, $\frac{dt}{t} \mapsto \frac{dt}{t}$.

$\boxed{\text{Exercise 1:}}$ Let $t = e^x$, $x \in \mathbb{R}$, in),

show that $\tilde{f}$ is $\hat{F}$ of some F (related to f)

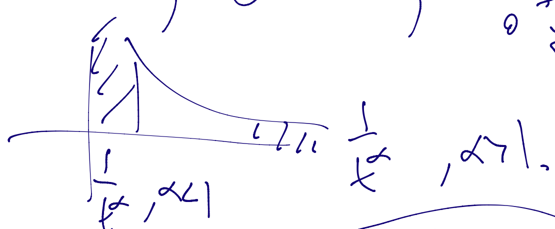
$(s \overset{?}{\leftrightarrow} \xi)$

E.g.: $f(t) = e^{-t}$. Does: $\int_0^\infty e^{-t}\, t^s \frac{dt}{t} = \tilde{f}(s)$ converge (to a hol'c function of s)?

$= \Gamma(s)$.

Near 0, $e^{-t} \approx 1$, $\int_0 1 \cdot t^{s-1}\, dt$ converges $\iff$ $(-\mathrm{Re}(s) < 1$

$0 < \mathrm{Re}(s)$.



Near ∞, $\int^\infty e^{-t} \cdot t^A < \infty$. wins.

Slower: $\int_\varepsilon^{1/\varepsilon} e^{-t} t^s \frac{dt}{t}$ $\xrightarrow{\varepsilon \to 0} \Gamma(s)$.

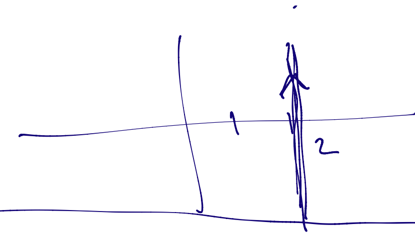
$\mathrm{Re}\, s > 0$

Again, $\tilde{f}(s) = \int_0^\infty f(t) \cdot t^s \frac{dt}{t}$.

Thm (Mellin inversion): $f \in C_c^\infty((0,\infty))$. smooth, cptly supported

$\implies \frac{1}{2\pi i} \int \tilde{f}(s)\, t^{-s}\, ds = f(t)$. $(t \in (0,\infty))$.

$$(2) \leftarrow \int_{2-i\infty}^{2+i\infty}$$



$\boxed{\text{Exercise:}}$ $\boxed{\text{Derive this from Fourier inversion formula}}$

pf: (Goldfeld-K): Fake pf 1: look at:

$$\frac{1}{2\pi i}\int_{(2)}\left[\int_0^\infty f(u)\,u^s\,\frac{du}{u}\right]t^{-s}\,ds \qquad t\cdot\delta_{u=t}.$$

Don't think, switch!

$$= \int_0^\infty f(u)\left[\underbrace{\frac{1}{2\pi i}\int_{(2)}\left(\frac{u}{t}\right)^s ds}\right]\frac{du}{u}$$

$$|\cdot| = \left(\frac{u}{t}\right)^{2=\mathrm{Re}(s)}.$$

Totally divergent, makes no sense.

Take 2: $\tilde{f}(s) = \int_0^\infty f(u)\,u^{s-1}\,du \overset{\text{by parts}}{=} 0 - \int_0^\infty f'(u)\,\frac{u^s}{s}\,du.$

$$\frac{1}{2\pi i}\int_{(2)}\left[-\int_0^\infty f'(u)\,\frac{u^s}{s}\,du\right]t^{-s}\,ds$$

$$= -\int_0^\infty f'(u)\left[\underbrace{\frac{1}{2\pi i}\int_{(2)}\left(\frac{u}{t}\right)^s\cdot\frac{1}{s}\cdot ds}\right]du \qquad \sigma + it = 2$$

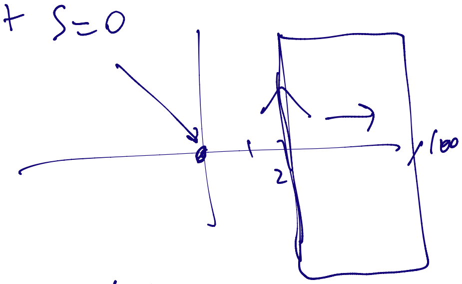
But nevermind, keep going. $\quad |\cdot| \approx \int_{-\infty}^{\infty}\left(\frac{u}{t}\right)^2\frac{1}{|\sigma+it|}\,dt \leftarrow \int\frac{1}{t} = \infty.$

interchange not justified

Claim: $I = \frac{1}{2\pi i}\int_{(2)} \left(\frac{u}{t}\right)^s \frac{ds}{s} = \begin{cases} 0, & \text{if } u<t \\ 1, & \text{if } u>t. \end{cases}$

merc with pole at $s=0$

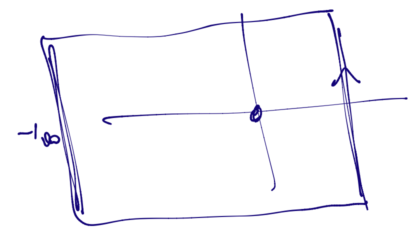
Case 1: $\frac{u}{t}<1$, pull contour right



$$\Rightarrow I = \frac{1}{2\pi i}\int_{(100)} \boxed{\left(\frac{u}{t}\right)^s}\frac{ds}{s}.$$

$$|\cdot| = \Big(\underbrace{\frac{u}{t}}_{<1}\Big)^{100} \to 0. \quad \text{as } 100\to\infty$$

Case 2: $\frac{u}{t}>1$, pull contour left



$$I = \operatorname{Res}_{s=0} + \cancel{\frac{1}{2\pi i}\int_{(-100)}\left(\frac{u}{t}\right)^s\frac{ds}{s}}^{\,0}$$

$$\left(\frac{u}{t}\right)^s\cdot\frac{1}{s}\cdot s\,\Big|_{s\to 0} = 1$$

$$= -\int_0^\infty f'(u)\cdot \mathbb{1}_{\{u>t\}}\,du = -\int_t^\infty f'(u)\,du$$

$$= -\left(\cancel{f(\infty)}^{\,0} - f(t)\right) = f(t).$$

Real pf: $\tilde{f}(s) = \int_0^\infty f(u)\,u^{s-1}du = -\int_0^\infty f'(u)\,\frac{u^s}{s}\,du = \int_0^\infty f''(u)\,\frac{u^{s+1}}{s(s+1)}\,du.$

Then: $\frac{1}{2\pi i}\int_{(2)}\left[\int_0^\infty \underbrace{f''(u)}_{\text{cpt supp}}\underbrace{\frac{u^{s+1}}{s(s+1)}}_{|\cdot|<\frac{1}{|1+s|^2}}du\right]t^{-s}\,ds \underset{\text{rigorously, converge absolutely.}}{=} \int_0^\infty f''(u)\left[\frac{1}{2\pi i}\int_{(2)}\left(\frac{u}{t}\right)^s\frac{1}{s(s+1)}\,ds\right]u\,du$

Exercise 3: $\frac{1}{2\pi i}\int_{(2)} \left(\frac{u}{t}\right)^s \underbrace{\frac{ds}{s(s+1)}}_{\text{Poles at } s=0,\ s=-1} = \begin{cases} 0, & \text{if } \frac{u}{t}<1 \\ 1-\frac{t}{u} & \text{if } \frac{u}{t}>1 \end{cases}$

$$= \int_0^\infty f''(u)\left[\left(1-\frac{t}{u}\right)\mathbb{1}_{\{u>t\}}\right]u\cdot du = \int_t^\infty f''(u)\,(u-t)\,du = -\int_t^\infty f'(u)\,du = f(t).$$

Rmk: No branch cuts! $\left(\frac{u}{t}\right)^s = e^{s\log\left(\frac{u}{t}\right)}$, $u,t\in\mathbb{R}_{>0}$. $\log_{\mathbb{R}}$.

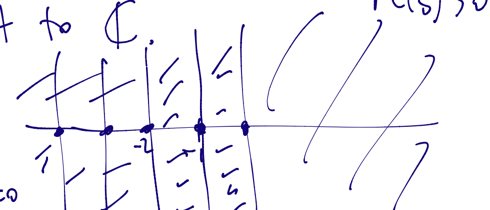
Back to
$$\Gamma(s) = \int_0^\infty e^{-t}\,t^s\,\frac{dt}{t} \underset{\text{by parts}}{=} \cancel{\left(e^{-t}\right)\left(\frac{t^s}{s}\right)\Big|_0^\infty} + \int_0^\infty \left(+e^{-t}\right)\frac{t^s}{s}\,dt$$
$$= \frac{1}{s}\int_0^\infty e^{-t}\,t^{s+1}\,\frac{dt}{t}$$
$$\Gamma(s) = \frac{1}{s}\cdot\Gamma(s+1).$$

Lemma: $\mathrm{Re}(s)>0$, $\Gamma(s+1) = s\cdot\Gamma(s)$, $\boxed{\text{Cor: } \Gamma(n+1)=n!}$

$$\Gamma(1) = \int_0^\infty e^{-t}\,\cancel{t}\,\frac{dt}{\cancel{t}} = 1. \qquad \Gamma(2) = 1\cdot\Gamma(1) = 1$$

$$\Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1, \qquad \Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2\cdot 1.$$

$\Gamma(s) = \frac{1}{s}\Gamma(s+1)$. Allows (mero'c) analytic cont to $\mathbb{C}$. $\mathrm{Re}(s)>0$.



If $\mathrm{Re}(s)>-1$ on $\mathrm{Re}(s)>-1$, $\Gamma(s) := \frac{1}{s}\Gamma(s+1)$ ↑ pole at $s=0$

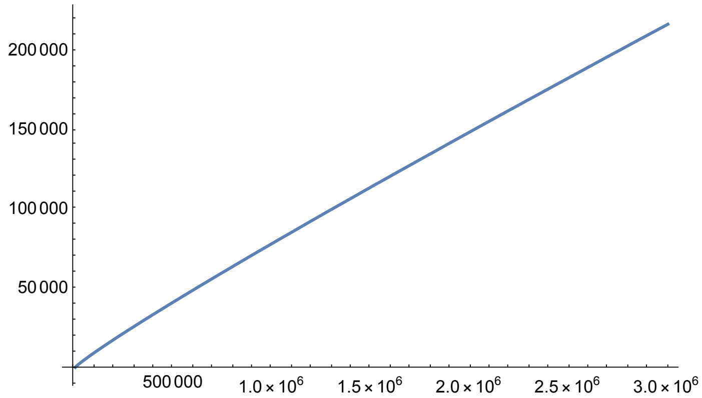
On $\mathrm{Re}(s)>-2$, $\Gamma(s) = \frac{1}{s}\cdot\underbrace{\Gamma(s+1)}_{\mathrm{Re}>-1}$. pole at $s+1=0$.

Rmk: Γ function is Not important. (Tate thesis) (& it's essential).

```
In[1]:= PrimePi[100]
Out[1]= 25
```

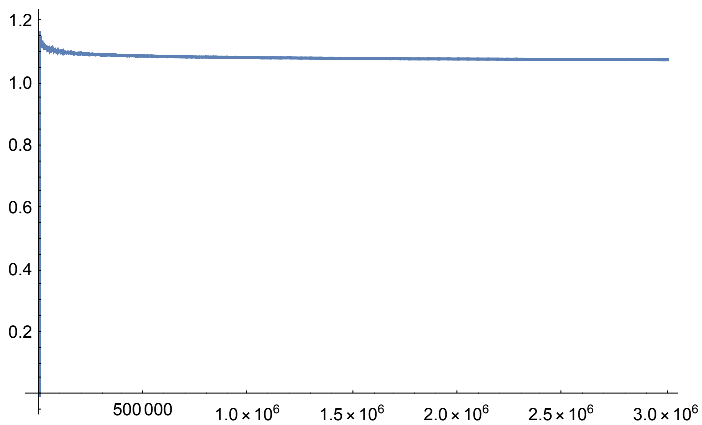
```
In[22]:= Plot[PrimePi[x], {x, 0, 3 000 000}, PlotPoints → 100, PlotRange → All]
```

Out[22]=



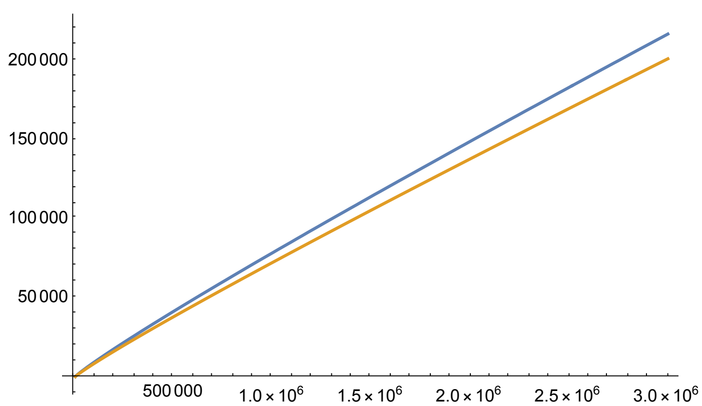

```
In[9]:= Plot[PrimePi[x] / (x / Log[x]), {x, 0, 3 000 000}, PlotPoints → 100, PlotRange → All]
```

Out[9]=



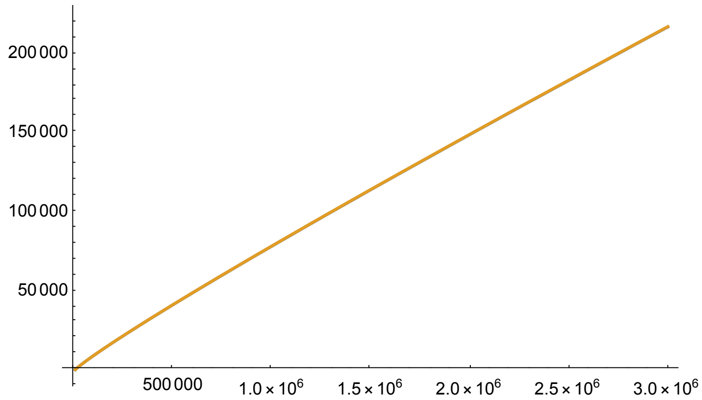

```
In[23]:= Plot[{PrimePi[x], x / Log[x]}, {x, 0, 3 000 000}, PlotPoints → 100, PlotRange → All]
```

Out[23]=



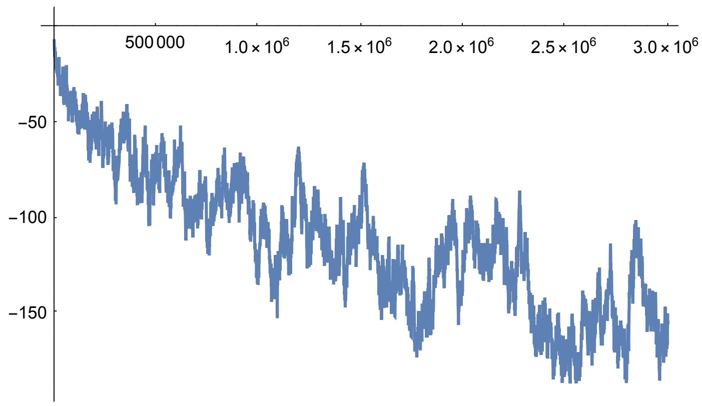
In[14]:=
```
Plot[{PrimePi[x], LogIntegral[x]}, {x, 0, 3 000 000}, PlotPoints → 100, PlotRange → All]
```

Out[14]=



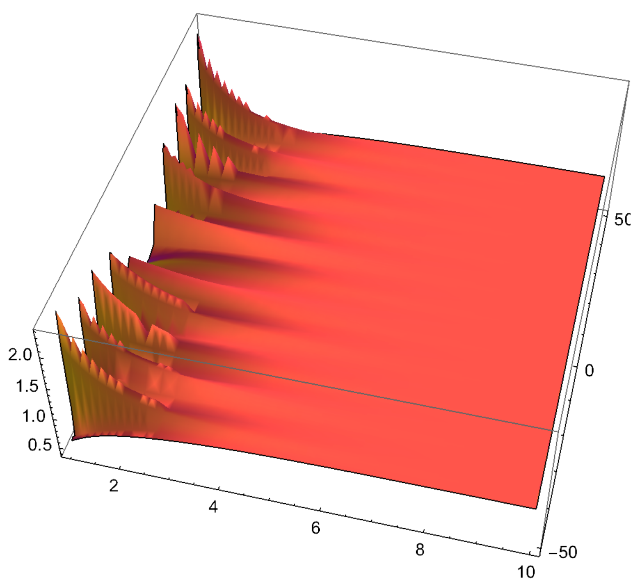
In[15]:=
```
Plot[PrimePi[x] - LogIntegral[x], {x, 0, 3 000 000}, PlotPoints → 100, PlotRange → All]
```

Out[15]=



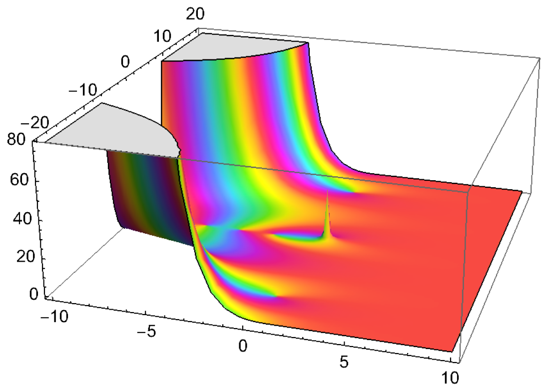
In[25]:=
```
ComplexPlot3D[Zeta[s], {s, 1 - I 50, 10 + I 50}, PlotRange → All]
```

Out[25]=



In[27]:= `ComplexPlot3D[Zeta[s], {s, -10 - I 20, 10 + I 20}]`

Out[27]=



In[21]:= `ComplexPlot3D[Zeta[s], {s, -10 - I 50, 10 + I 50}, PlotRange → All]`

Out[21]=