Last time: Solved Basel Problem $\sum \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}$, modulo.

$F(z) = \pi \cot \pi z \overset{?}{=} \sum_{n\in\mathbb{Z}} \frac{1}{z+n} := \frac{1}{z} + \sum_{n\geq 1} \frac{2z}{z^2-n^2}$ ← not summable ← hol'c near 0. ← $\frac{1}{n^2}$

$\pi \frac{\cos \pi z}{\sin \pi z}$

$\cos\pi z = 1 + o()$

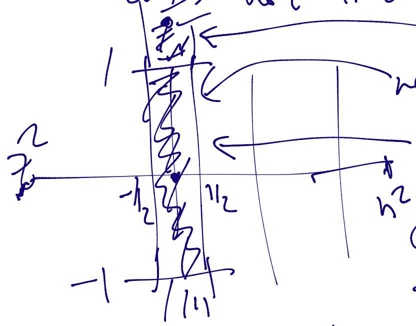
$\sin \pi z = \pi z + o()$.

• $F(z) = F(z+1)$. • near 0, $F(z) = \frac{1}{z} + F_0(z)$ ← hol'c near $z=0$.

Let $\Delta(z) := \pi \cot \pi z - \sum \frac{1}{z+n}$, $\Delta(z) = \Delta(z+1)$.

Is hol'c at 0. & at all $\mathbb{Z}$, ⇒ Entire. (if Δ bdd ⇒ Const)

(Δ odd) ⇒ $\Delta = 0$).



need bdd: $y>1$.

bdd since hol'c.

Claim: $F(z)$ & $\sum \frac{1}{z+n}$ both bounded for $|x| \leq \frac{1}{2}$, $y \geq 1$.

$$|F(z)| = |\pi \cot \pi z| = \left|\pi \frac{\cos \pi z}{\sin \pi z}\right| = \left|\pi i \frac{(e^{i\pi z} + e^{-i\pi z})}{(e^{i\pi z} - e^{-i\pi z})} \frac{e^{-i\pi \bar{z}}}{e^{-i\pi \bar{z}}}\right| = \left|\pi i \frac{e^{i\pi(2iy)} + e^{-\pi i(2x)}}{e^{-2\pi y} - e^{-2\pi i x}}\right|.$$

$$\leq \pi \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}} \leq 10.$$

Now check if other part bdd? $y \geq 1$, $|x| \leq \frac{1}{2}$.

$$\left|\frac{1}{z} + \sum_{n\geq 1} \frac{2z}{z^2-n^2}\right| \leq C + C\sum_{n\geq 1} \frac{2|z|}{|z|^2+n^2} \leq C + C\int_0^\infty \frac{r}{r^2+x^2}\,dx$$

Claim: $\int_0^\infty \frac{r}{r^2+x^2}\,dx < C$ indep of r.

Pf: $x \mapsto rx$ $\quad = \int_0^\infty \frac{r}{r^2+r^2x^2}\, r\,dx = \int_0^\infty \frac{1}{1+x^2}\,dx < C$.

So: $\left|\sum \frac{1}{z+n}\right| < C$ on $|x| \leq 1/2$, $|y| \geq 1$, & $|\pi \cot \pi z| < C$

⇒ Δ bdd ⇒ Const ⇒ $0 = \Delta$.

Recap: Knowing where zeros of $\sin z$ were, we gave a formula $\sin z \overset{\checkmark}{=} z \cdot \prod_{n\in\mathbb{Z}\setminus\{0\}} \left(1-\frac{z}{\pi n}\right)$

Thm: (Weierstraß): Let $f \not\equiv 0$ be entire, & say it has zeros at $\{a_n\}$ (w/ mult.) $\Rightarrow$ a_n's have no accumulation pt, $\Leftrightarrow$ $|a_n| \to \infty$. Conversely:

① Given any seq $\{a_n\}$ with $|a_n| \to \infty$, $\exists$ some entire f having exactly these zeros & no others.

② Moreover, any other such is of the form fe^{g} ← entire.

Pf ②: If f_1 & f_2 have same zeros (& mult) $\Rightarrow$ $\frac{f_1}{f_2}$ is bdd ($\Rightarrow$ removable) at each zero $\Rightarrow$ entire, nonzero, so $\exists$ "$\log \frac{f_1}{f_2}$" $= g$, so $e^{g} = \frac{f_1}{f_2} \Rightarrow f_1 = f_2 e^{g}$.

To prove ①: "Naive"/obvious thing: $z^{m} \prod_n \left(1-\frac{z}{a_n}\right) \overset{?}{=} f(z)$.

$0 \le \#\{a_n = 0\}$

But!!! No convergence! a_n's $\to \infty$ however slowly they like

Weierstrass: "canonical factors" Let $E_0(z) = 1-z$, $\underset{k\geq 1}{E_k(z)} = \underset{(1-z)}{\wedge} e^{\underbrace{z+\frac{z^2}{2}+\frac{z^3}{3}+\dots+\frac{z^k}{k}}}$.

k terms of $-\log(1-z)$

$(1-z)e^{-\log(1-z)} = 1$,

Lemma: If $|z| \leq \frac{1}{2}$, then $(k \geq 1)$

$$|E_k(z)-1| \leq 10 \cdot |z|^{k+1}. \checkmark$$

Pf: $|z| \leq \frac{1}{2}$, $1-z \neq 0$, $\exists \log(1-z)$ s.t. $(1-z) = e^{\log(1-z)}$.

So $E_k(z) = (1-z)e^{\cdots} = e^{\overbrace{\log(1-z)+z+\frac{z^2}{2}+\frac{z^3}{3}+\dots+\frac{z^k}{k}}^{w}}$.

$$\left|\log(1-z)+z+\dots+\frac{z^k}{k}\right| = \left|-\frac{z^{k+1}}{k+1} - \frac{z^{k+2}}{k+2} - \cdots\right|$$

- $\leq \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \cdots \leq 1$.
- $\leq |z|^{k+1} \left|\frac{1}{k+1} + \frac{1}{2^{k+2}} + \frac{1}{2^{k+3}} + \cdots\right| \leq 2|z|^{k+1}$.

$$|w| < 1 \Rightarrow |e^w - 1| \leq 5|w| \leq 10|z|^{k+1}$$

Weierstrass: $|E_k(z)-1| \leq 10 \cdot |z|^{k+1}$ (if $|z| \leq 1/2$).

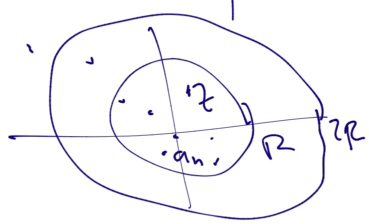
$E_k(z) = (1-z)e^{z+\dots+\frac{z^k}{k}}$

("diagonalization").

Let: $f(z) := z^m \prod_n E_n\left(\frac{z}{a_n}\right)$

Claim: f converges $\Rightarrow$ has exactly the zeros it should.

: Fix R, let $|z|<R$. There are fin many $|a_n|<2R$



$$f(z) = z^m \overbrace{\prod_{n:\,|a_n|<2R}}^{\text{finite}} E_n\left(\frac{z}{a_n}\right) \prod_{n:\,|a_n|\geq 2R} \underbrace{E_n\left(\frac{z}{a_n}\right)}.$$

$\left|\frac{z}{a_n}\right| \leq 1/2$

on $\left|\frac{z}{a_n}\right| \leq 1/2$, $\Rightarrow \left|E_n\left(\frac{z}{a_n}\right) - 1\right| \leq 10 \left|\frac{z}{a_n}\right|^{n+1} \leq 10 \cdot \frac{1}{2^{n+1}}$ summable.

$\Rightarrow \prod$ converges & vanishes $\Leftrightarrow$ one term vanishes.

True for any R, hence f is entire & has right zeros

Rmk: Then can get any poles & zeros you want (take a ratio).

Need to know "functions of finite order".

Def: If f entire & $\exists C, \exists k : \forall z,$ $|f(z)| \leq C e^{C \cdot |z|^k}$

$\Rightarrow f$ has order $\leq k$.

$k_f := \inf_{f \text{ has order } k} k$ = order of f.

E.g.: • $\sin z$ order $= 1$ $\qquad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

• e^{z^2} order $= 2$.

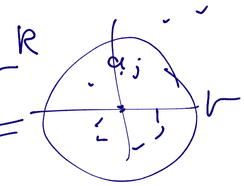
• e^{e^z} order $= \infty$. ← function of infinite order

• $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n)!}$ "$=$" $\cos z^{1/2}$ order $= \frac{1}{2}$.

• $e^{e^{iz}}$ still ~~order~~ $= \infty$.

"Thm": Let f have order $\le k$, & zeros $0, 0, \dots, 0$, $a_1, a_2, \dots$ ← nonzero zeros.

Then: • $n(r) := \#\{|a_j| < r\} \le C \cdot r^k$

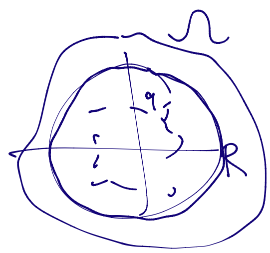


• $\displaystyle\sum_n \frac{1}{|a_n|^s} < \infty \quad \forall\, s > k$.

Ingredients in pf:

Jensen's Formula: Let $f : \Omega \to \mathbb{C}$ hol'c, $\supset \overline{D_R(0)}$ & has zeros $0 \neq a_1, \dots, a_N \in D_R(0)$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| \, d\theta \overset{\star}{=} \log |f(0)| - \sum_{j=1}^{N} \log \left(\frac{|a_j|}{R} \right).$$

Pf: 

Let $g(z) = \dfrac{f(z)}{(z-a_1)\cdots(z-a_N)}$

has no zeros in D_R & has removable singularities. $\Rightarrow$ g holo.

$\Rightarrow$ $f(z) = (z-a_1)\cdots(z-a_N)\, g(z)$, $g \neq 0$ on $\overline{D_R}$

$\Rightarrow$ $\log|f(z)| = \log|z-a_1| + \cdots + \log|z-a_N| + \log|g(z)|$.

Will prove piece by piece. $g \neq 0 \Rightarrow \exists\, h = \log g$, $e^h = g$.

$$\frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\, d\theta = \log|g(0)|.$$

$|g| = e^{\mathrm{Re}\, h}$, $\underbrace{\log|g|}_{\text{harmonic}} = \mathrm{Re}\, h$. $\Rightarrow$ Mean Value Prop.

$|\alpha| = \frac{|a_1|}{R} < 1$.

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|R\left(e^{i\theta} - \frac{a_1}{R}\right)\right| d\theta = \log R + \frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - \alpha|\, d\theta$$

$\theta \mapsto -\theta$. $\overbrace{\log|1 - e^{-i\theta}\alpha|}^{|e^{-i\theta}|=1}$

$$= \log R + \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \log|1 - e^{i\theta}\alpha|\, d\theta}_{= \log|\varphi(0)| = \log 1 = 0.}$$

But $\varphi(z) := 1 - z\alpha$ has no zeros in D_1, zero at $|z| = \left|\frac{1}{\alpha}\right| > 1$.

$\Rightarrow$ $\exists\, \psi = \log\varphi$ in $\overline{D_1}$, $\log|\varphi| = \mathrm{Re}\,\psi :=$ harmonic

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta = \log R + \cdots + \log R + \log|g(0)|$$

$$\boxed{\log|f(0)| = \log|a_1| + \cdots + \log|a_N| + \log|g(0)|.}$$

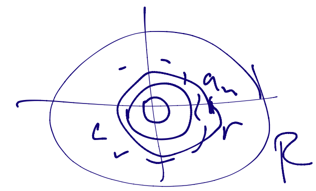
$+ \log| \ldots$

$\log|f(0)| - \log|f(0)|.$

Jensen's Formula:

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\,d\theta = \log|f(0)| - \sum_{j=1}^{N}\log\left|\frac{a_j}{R}\right| \quad ✓$$

Lemma: $n(r) = \#\{\alpha : |a_n| < r\}$



$$\int_0^R n(r)\frac{dr}{r} = \sum_{j=1}^{N}\log\left|\frac{R}{a_j}\right|$$

Pf: let $\chi_j(r) = \begin{cases} 1, & |a_j| > r \\ 0, & \end{cases}$, $\quad n(r) = \sum_{j=1}^{N}\chi_j(r).$

$$\sum_{j=1}^{N}\int_0^R \chi_j(r)\frac{dr}{r} = \sum_{j=1}^{N}\int_{|a_j|}^{R} 1\cdot\frac{dr}{r}$$

Cor: $$\int_0^R n(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\,d\theta - \log|f(0)|.$$

Thm: If f has order $\le k$ then $\boxed{\log|f(Re^{i\theta})| \le CR^k + C}$ $\boxed{|f(z)| \le Ce^{C|z|^k}}$

- $n(r) \le C\cdot r^k$ ✓
- $\displaystyle\sum_{a_n\neq 0}\frac{1}{|a_n|^s} < \infty \qquad s > k.$

Pf: Dyadic $\quad n(r)\cdot c \le \int_r^R n(r)\,dr \le \int^R n(r)\frac{dr}{r} \le C + C\cdot R^k.$

decomp $\underbrace{\int_{R/2}\ \cdots}$

$z\cdot e^{z} = e^{z\log z}$ has order 1 but not order ≤ 1.

Finite # a_n's $|a_n|<1$.

$$\underbrace{\sum_{|a_n|<1}\frac{1}{|a_n|^s}} + \sum_{|a_n|\ge 1}\frac{1}{|a_n|^s}.$$

$$\le C + \sum_{\ell=0}^{\infty}\ \sum_{2^{\ell}\le |a_n|\le 2^{\ell+1}} \frac{1}{|a_n|^s} \le C+\sum_{\ell=0}^{\infty}\frac{1}{2^{\ell s}}\cdot \overbrace{n(2^{\ell+1})}^{\le\, C(2^{\ell+1})^R}$$

$$\le C+\sum_{\ell=0}^{\infty}\underbrace{\frac{1}{2^{\ell s}}\cdot 2^{\ell R}}_{s>R} \le C<\infty.$$

Hadamard Thm: If f has order $\le R$ & zeros $a_n \ne 0$, then let $\ell=\lfloor R\rfloor$

$$f(z)=e^{P(z)}z^m\prod_n E_\ell\left(\frac{z}{a_n}\right)$$

Poly of degree $\leq l$. A uniform over n.