New topic: History: Leibniz solved problem Huygens:

Compute sum reciprocals of $\triangle$ #s.

1, 3, 6, 10, 21

$\sum_1^n i = \frac{n(n+1)}{2}$.

$$S = \frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{21}+\dots = \sum \frac{2}{n(n+1)} = 2\sum \frac{1}{n(n+1)}$$

$\frac{1}{n} - \frac{1}{n+1}$

$$= 2\left[\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\dots\right] = 2.$$

$1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+$

1690s?

Next Q: Sum reciprocals of $\square$ #s?

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\dots < 2.$$

$= 1.6449\ldots$!

Jakob Bernoulli? Johann? Leonhard Euler. 1720s.

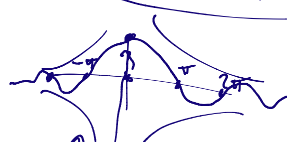
Daniel 1727, St. Petersburg.

Basel Problem. 1734

Euler: $\left(e^{i\theta} = \cos\theta + i\sin\theta\right)$ $(e^{i\pi}+1=0)$.

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots$$



Obs: If poly $P \in \mathbb{C}[x]$ & $P(0)=1$, & P has roots $\alpha_1,\dots,\alpha_d \neq 0$ (deg d). $\Rightarrow P(x) = \left(1-\frac{x}{\alpha_1}\right)\left(1-\frac{x}{\alpha_2}\right)\cdots\left(1-\frac{x}{\alpha_d}\right)$.

$d+1$ coefficients

Then: unrigorous

$$\frac{\sin z}{z} = \left(1-\frac{z}{\pi}\right)\left(1+\frac{z}{\pi}\right)\left(1-\frac{z}{2\pi}\right)\left(1+\frac{z}{2\pi}\right)\left(1-\frac{z^2}{9\pi^2}\right)\left(1-\frac{z^2}{16\pi^2}\right)\cdots$$

$$1-\frac{z^2}{3!}+\frac{z^4}{5!}-\dots = \frac{\sin z}{z} = \left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{4\pi^2}\right)\left(1-\frac{z^2}{9\pi^2}\right)\left(1-\frac{z^2}{16\pi^2}\right)\cdots$$

Expand Look at z^2 term!!!

Looks at z^4 term

Finds

$\frac{1}{1}+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\cdots$

$\zeta(4)=\sum\frac{1}{n^4}=\pi^{4}\cdot$ Bernoulli #'s

$$-\frac{1}{6}. = -\frac{1}{\pi^2}-\frac{1}{4\pi^2}-\frac{1}{9\pi^2}-\frac{1}{16\pi^2}-\cdots$$

$$\boxed{\frac{\pi^2}{6} = \frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots} = \zeta(2).$$

$\sum\frac{1}{n^{2k}} = \checkmark$ Euler. $\zeta(3)=\sum\frac{1}{n^3}=\frac{1}{1}+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\frac{1}{125}+\cdots$

$=???\ \notin\mathbb{Q}$ (Apéry 1970s).

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}. \quad \left(\sim\int_1^\infty\frac{1}{x^s}dx\right)$$

"Riemann Zeta Function

Dirichlet Series defined by Euler".

$$\zeta(s)=\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\cdots+\frac{1}{12^s}+\cdots\right) = \left(1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{8^s}+\frac{1}{16^s}+\cdots\right)$$

$\frac{1}{2^{2s}\cdot 3^s}$ geom $\frac{1}{1-\frac{1}{2^s}}$

$$\times\left(1+\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{27^s}+\frac{1}{81^s}+\cdots\right)$$

$$\times\left(1+\frac{1}{5^s}+\frac{1}{5^{2s}}+\frac{1}{5^{3s}}+\cdots\right)$$

$$\times\left(1+\frac{1}{7^s}+\frac{1}{49^s}+\cdots\right)$$

$\boxed{\mathrm{Re}(s)>1}$ (σ)

$\boxed{z=x+iy \Rightarrow s=\sigma+it}$

Unique factorization of $n\in\mathbb{N}$ as product of prime powers.

$\left|\frac{1}{n^{\sigma+it}}\right| = \left|\frac{1}{e^{(\sigma+it)\log n}}\right| = \frac{1}{n^\sigma}.$

$\boxed{1737;}$ "Euler product formula";

$$\sum_{n\geq 1}\frac{1}{n^s} = \prod_p\left(1-p^{-s}\right)^{-1}$$

$\boxed{\text{Thm: } \sum_p\frac{1}{p}=\infty.}$ $\Rightarrow \infty$ly many primes (Euclid)

$$\frac{\pi^2}{6}=\zeta(2)=\left(\frac{1}{1-\frac{1}{2^2}}\right)\left(\frac{3^2}{3^2-1}\right)\left(\frac{5^2}{5^2-1}\right)\left(\frac{7^2}{7^2-1}\right)\cdots$$

$\frac{2^2}{2^2-1}$

Note: $\boxed{\sum\frac{1}{n}=\infty} = \zeta(1).$

"Birth of Analytic Number Theory"

$$\Rightarrow \boxed{\sqrt{6}\cdot\frac{2}{\sqrt{3}}\cdot\frac{3}{\sqrt{8}}\cdot\frac{5}{\sqrt{24}}\cdot\frac{7}{\sqrt{48}}\cdot\frac{11}{\sqrt{120}}\cdot\frac{13}{\sqrt{168}}\cdots = \pi}$$

$\zeta(s)\to\infty$ as $s\to 1$. Look at $\log\zeta(s) = -\sum_p\log(1-p^{-s})$

Knows: $-\log(1-x) = \underbrace{x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}\cdots}$ $|x|\leq\frac{1}{2}$

∞ as $s\to 1$ $= \underbrace{\sum_p\frac{1}{p^s}} + \underbrace{O\left(\sum_{n=2}^{\infty}\frac{1}{n^{2s}}\right)}_{O(1). \text{ as } s\to 1.}$

$$\frac{d}{dx} = +\frac{1}{1-x} \overset{??}{=} 1+x+x^2+x^3$$

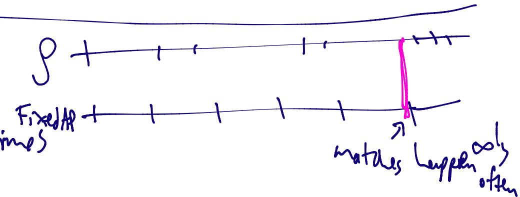
1837: Dirichlet $\sum_{\substack{p \equiv a(q) \\ (a,q)=1}} \frac{1}{p} = \infty$. ($\Rightarrow \infty$)

Eg: $\Rightarrow$ ∞ly many primes end in ...777. $(777, 10^3) = 1$.

1859: Riemann Memoir on ζ (RH)

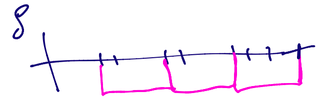
1896: Hadamard - de la Vallée Poisson Prove Gauss Conj, Prime Number Theorem

3 Statements about Primes in Progressions:



(1) Dirichlet: $\forall$ Fixed AP $\exists$ ∞ly many Primes

(2) Green-Tao '04?: Inside Primes, $\exists$ k-term APs



$\Uparrow$ Primes not consecutive, spacing same

Erdös-Turan Conj: For Any set of $S \subset \mathbb{N}$ with $\sum_{s \in S} \frac{1}{s} = \infty \Rightarrow \exists$ k-term AP.

Bloom-Sisask 2020: $k=3$. If S has no 3-term AP $\Rightarrow \#\{s \in S \mid s < X\} \ll \frac{X}{\log^{1+\varepsilon} X}$. $\Rightarrow \sum \frac{1}{s} < \infty$.

(3) Shiu '96?: $\forall$ Fixed AP $\exists$ k consecutive primes $\equiv a(q)$.

i.e. in the table of primes, somewhere is a list of $k = 1000$ primes in a row!!! (Consecutive) all ending in 777.

Make this stuff rigorous. Q: What does it mean for $\prod(1+a_n)$ to converge?

Def: Given $\{a_n\} \subset \mathbb{C}$, we say $\prod_{n=1}^{\infty}(1+a_n)$ converges iff $A_N = \prod_{n=1}^{N}(1+a_n) \in \mathbb{C}$ partial products converge.

Lemma: If $\sum_n |a_n| < \infty$ then $\prod_{n=1}^{\infty}(1+a_n)$ converges.

$\boxed{\sum |a_n| < \infty \Rightarrow a_n \to 0 \text{ Assume } |a_n| < \frac{1}{2}}$

Pf: Look at $\log A_N = \sum_{n=1}^{N} \log(1+a_n)$.

$$-\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots$$

unif convergence $\boxed{\text{for } |z| \le \frac{1}{2}}$

$\frac{1}{2}$, 1

$$|\log(1-z)| \le |z| \cdot 2.$$

But $\sum_{n=1}^{N} |\log(1+a_n)| \le 2 \cdot \sum_{n=1}^{N} |a_n| < \infty$.

$\Rightarrow \sum_{n=1}^{\infty} \log(1+a_n)$ converges, exp is cont. $\Rightarrow \exp(\log A_N)$ has a limit ✓.

Moreover, if $\sum |a_n| < \infty$ & $\prod_{n=1}^{\infty}(1+a_n) = 0 \Rightarrow$ some $a_n = -1$, i.e. some $1+a_n = 0$.

Pf: $\exp(\log A_N) \neq 0$.

Prop: Let $\{F_n(z)\}_{n=1}^{\infty}$ hol'c on Ω. Assume $\exists\, c_n$ (indep of z) s.t. $\forall z \in \Omega$ $|\underbrace{F_n(z) - 1}_{a_n}| < c_n$ & $\sum c_n < \infty$. Then: $\prod_{n=1}^{\infty} F_n(z)$ converges unif on cpta to hol'c F on Ω.

Pf: Fix $z \in \Omega$, let $a_n(z) := F_n(z) - 1$. Then $|a_n(z)| < c_n$ uniform over $K \overset{cpt}{\subset} \Omega$.

$\Rightarrow G_N(z) = \prod_{n=1}^{N} F_n(z)$ converges uniformly on K to $G_N(z) \to F(z) =$ hol'c on Ω.

Claim: Moreover if No $F_n(z) = 0 \Rightarrow \frac{F'}{F}(z) = \sum_n \frac{F_n'}{F_n}$. $F = \prod_n (F_n)$

Pf: For each N, $\frac{G_N'}{G_N} = \sum_{n=1}^{N} \frac{F_n'}{F_n}$. $G_N \xrightarrow{\text{unif on cpt}} F \Rightarrow G_N' \xrightarrow{\text{unif on cpta}} F'$, & $\frac{1}{G_N}$ unif bdd on K.

$F \neq 0 \Rightarrow |G_N(z)| \geq \varepsilon$ unif on K.

$\Rightarrow \frac{G_N'}{G_N} \xrightarrow{\text{unif conv on } K} \frac{F'}{F}$.

Justify Euler: Claim: $\boxed{\pi \cot \pi z = \sum_{n \in \mathbb{Z}} \frac{1}{z+n}}$ $(z \notin \mathbb{Z})$. doesn't converge!

$\cot = \frac{\cos}{\sin}$

Claim $\Rightarrow Q(z) = \frac{\sin \pi z}{\pi} \overset{?}{=} z \prod_{n=1}^{\infty} \underbrace{\left(1 - \frac{z^2}{n^2}\right)}_{F_n(z)} = P(z)$. Converge unif? $|F_n(z) - 1| = \frac{|z|^2}{n^2} \leq \frac{c}{n^2}$ If $z \in K$. Series converges.

$P(z)$ converges $\Rightarrow$

$$\frac{P'}{P}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \overbrace{\frac{-2z}{n^2}}^{F_n'} \cdot \overbrace{\frac{1}{1 - \frac{z^2}{n^2}}}^{\frac{1}{F_n}} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

$$\frac{-2z}{n^2 - z^2}$$

$$\frac{Q'}{Q}(z) = \frac{\frac{1}{\pi}\cos \pi z \cdot \pi}{\frac{1}{\pi}\sin \pi z} = \pi \cot \pi z$$

$\sum_{|n| < N} \;=\; \frac{1}{z} + \sum_{n=1}^{N} \underbrace{\frac{1}{z+n} + \frac{1}{z-n}}_{\text{group}}$.

$$\left(\frac{P}{Q}\right)' = \frac{Q\cdot P' - P\cdot Q'}{Q^2} = \frac{P'}{Q} - \frac{P}{Q}\cdot\frac{Q'}{Q} = \frac{P}{Q}\underbrace{\left(\frac{P'}{P} - \frac{Q'}{Q}\right)}_{=0} = 0.$$

$P = c\cdot Q$. Divide by z & send $z \to 0$ $\Rightarrow c = 1$.