Last time: Schwarz Lemma: $f:\mathbb{D}\to\mathbb{D}$, hol'c (not nec. bij!), $f(0)=0$, non-const.

Then (i) $\forall z\in\mathbb{D}$, $|f(z)|\le|z|$ (ii) If $\exists z_0\in\mathbb{D}: |f(z_0)|=|z_0| \Rightarrow f(z)=e^{i\theta}z$.

(iii) $|f'(0)|\le 1$ & (iv) If $|f'(0)|=1 \Rightarrow f(z)=e^{i\theta}z$.

Let $\mathrm{Aut}\,\mathbb{D}=\{f:\mathbb{D}\to\mathbb{D} \text{ hol'c, bij}\}$ (gp under $\circ$). $(g\circ f)(z)=z$

$\boxed{g'(f(z))\cdot f'(z)=1,\quad f(z)=w,\ g(w)}$

Recall Blaschke factors: $\psi_\alpha(z):=\frac{z-\alpha}{1-\bar\alpha z}=\begin{pmatrix}1&-\alpha\\-\bar\alpha&1\end{pmatrix}\circ z$. (Book uses $\frac{\alpha-z}{1-\bar\alpha z}$).

(This is not an involution, $\psi_\alpha^{-1}=\begin{pmatrix}1&\alpha\\ \bar\alpha&1\end{pmatrix}=\psi_{-\alpha}$.)

This is an automorphism $\iff |\alpha|<1$. Pole at $z=\frac{1}{\bar\alpha}\notin\mathbb{D}$ $\Leftrightarrow$ $\frac{1}{|\alpha|}>1$

$\boxed{\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}\underset{PGL_2}{=}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}}$

$\boxed{\begin{pmatrix}d&-b\\-c&a\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}ad-bc&0\\0&ad-bc\end{pmatrix}}$

Thm: $\mathrm{Aut}\,\mathbb{D}=\mathbb{D}\times S^1$

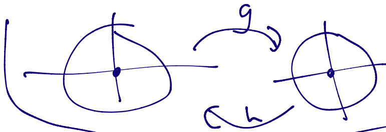
i.e. $f \Rightarrow \exists \alpha\in\mathbb{D},\ \theta\in\mathbb{R}/2\pi\mathbb{Z}$: $f(z)=e^{i\theta}\psi_\alpha(z)$.

$\boxed{\psi_\alpha(0)=-\alpha}$

Pf: Given $f\in\mathrm{Aut}\,\mathbb{D}$. Let $\alpha=f^{-1}(0)\in\mathbb{D}$. Let $g:=f\circ\psi_{-\alpha}\in\mathrm{Aut}(\mathbb{D})$.

$g(0)=f(\psi_{-\alpha}(0))=f(\alpha)=0$, $\overset{\text{Schw}}{\Rightarrow}\forall z\in\mathbb{D}$, $|g(z)|\le|\underset{h(w)}{z}|$.

Let $h=g^{-1}$. Then $h(0)=0$. $\Rightarrow \forall w\in\mathbb{D}$, $|z|=|h(w)|\le|w|=|g(z)|$.



$\Rightarrow |g(z)|=|z| \overset{\text{Schw}}{\Rightarrow} g(z)=e^{i\theta}z$.

$(f\circ\psi_{-\alpha})(z)=e^{i\theta}z$.

$\boxed{f\circ\psi_{-\alpha}\circ\psi_\alpha = e^{i\theta}\psi_\alpha(z)}$

Claim: $\mathrm{Aut}\,\mathbb{D}=(P)SU(1,1)$.

What is $SO(n,m)$, $SU(n,m)$? (n: $+1$, m: -1)

$Q_2(x,y)=x^2+y^2$

$Q(x,y)=x^2-y^2$

$Q(x)=x^2$.

Quadratic forms$/\mathbb{R}$: $Q:V_{\mathbb{R}}\to\mathbb{R}$ ← homogeneous poly of deg 2.

$Q(\alpha x)=\alpha^2Q(x)$, $\alpha\in\mathbb{R}$.

Bilinear form: $B : V \times V \to \mathbb{R}$ $B(\alpha x + \beta x', y) = \alpha B(x,y) + \beta \cdot B(x',y)$ & same for y.

$Q(x) := B(x,x)$.

$(x+y)^2 - (x-y)^2 = x^2 + 2xy + y^2 - (x^2 - 2xy + y^2) = 4xy$.

$B(x,y) := \frac{1}{4}\left[Q(x+y) - Q(x-y) \right]$.

$B_2(\cdots$

$\boxed{B_0(x,y) = x \cdot y.}$ $B_1((x,y),(x',y')) = x x' - y y' = (x' \; y')\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Symmetric matrix $M = \left(a_{ij} = a_{ji} \right)$, $Q\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x^t M x$.

half-Hessian / gram matrix. $B(\vec{x}, \vec{x}') = x'^t M x$.

(a_{ij}).

$Q(x,y) = x^2 - y^2$ half-Hessian: $\frac{1}{2}\begin{pmatrix} \partial_{xx} Q & \partial_{xy} Q \\ \partial_{yx} Q & \partial_{yy} Q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Over $\mathbb{R}$, Symmetric matrix diagonalizable $M \sim \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ $Q(v_1) = \lambda_1 v_1$.

$v_1' = \frac{1}{\sqrt{|\lambda_1|}} v_1$.

Classification of Quadratic forms $/\mathbb{R} = (p, n, z)$

Signature $\to$ $p = \#{+1}$'s, $n = \#{-1}$'s, $z = \#0$'s.

$\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & 0 \end{pmatrix}$ signature Q: $(2,1,1)$.

$Q(x,y,z,w) = x^2 + y^2 - z^2$

Over $\mathbb{C}$: $\overline{M}^t = M^* = M = \left(a_{ij} = \overline{a_{ji}} \right)$ Hermitian matrix.

Diagonalizable: (Signature: ($\underline{\underline{+1}}$'s, ~~$-1$'s~~, $\underline{\underline{0}}$'s))

& real eigenvalues.

Sesquilinear form: $B(\underset{\mathbb{C}^n}{x}, \underset{\mathbb{C}^n}{y}) \Rightarrow B(\alpha x+\beta x', y) = \alpha B(x,y) + \beta B(x',y)$.

$1\frac{1}{2}$

linear in 1st var, conj-linear in 2nd. $B(x, \alpha y+\beta y') = B(x,y)\cdot\bar{\alpha} + B(x,y')\cdot\bar{\beta}$.

$B(x,y) = \bar{y}^t M x$, Hermitian form $Q(x) = B(x,x)$.

$SU(1,1) := \{ g \in SL_2\mathbb{C} :$ stabilize Hermitian form of sgn$(1,1)\}$.

$$Q(\underset{\mathbb{C}}{z}, \underset{\mathbb{C}}{w}) = |z|^2 - |w|^2 = (\bar{z}\ \bar{w})\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}z\\w\end{pmatrix}$$

If g stabilizes Q: $Q\left(g\begin{pmatrix}z\\w\end{pmatrix}\right) = Q\begin{pmatrix}z\\w\end{pmatrix}\ \forall \begin{pmatrix}z\\w\end{pmatrix} \in \mathbb{C}^2$.

"automorphism group of Q",

$$(\bar{z}\ \bar{w})\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}z\\w\end{pmatrix} = (\bar{z}\ \bar{w})\bar{g}^t\begin{pmatrix}1&0\\0&-1\end{pmatrix}g\begin{pmatrix}z\\w\end{pmatrix}$$

Write $g = \begin{pmatrix}a&b\\c&d\end{pmatrix} \Longrightarrow \begin{pmatrix}\bar{a}&\bar{c}\\\bar{b}&\bar{d}\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}$

$$= \begin{pmatrix}\bar{a}&-\bar{c}\\\bar{b}&-\bar{d}\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}|a|^2-|c|^2 & \bar{a}b-\bar{c}d\\ \bar{b}a-c\bar{d} & |b|^2+|d|^2\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} |a|^2 - |c|^2 = 1 \\ |b|^2 - |d|^2 = -1 \\ \bar a b - \bar c d = 0 \end{array}\right\} \qquad a\bar a b = \bar c d a \qquad (ad - bc = 1).$$

$$|a|^2 b = \bar c(1 + bc) = \bar c + b|c|^2.$$

$$b(|a|^2 - |c|^2) = \bar c \Rightarrow b = \bar c.$$

$$\bar a \underset{\bar c}{b} = \bar c d \Rightarrow \bar a = d, \quad g = \begin{pmatrix} a & b \\ \bar b & \bar a \end{pmatrix} \in SU(1,1).$$

$$\& \ \det g = 1 = |a|^2 - |b|^2 \iff \frac{1}{|b|^2} + \cancel{\frac{|b|^2}{1}} = \frac{|a|^2}{|b|^2} \Rightarrow \left|\frac{a}{b}\right|^2 > 1.$$

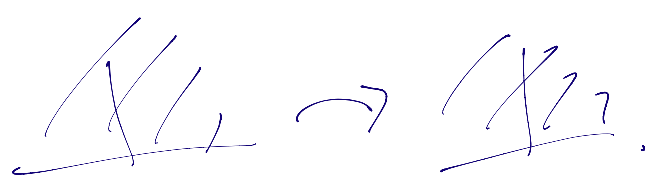
$$g \circ z = \begin{pmatrix} a & b \\ \bar b & \bar a \end{pmatrix} \circ z = \frac{az + b}{\bar b z + \bar a} = \frac{a}{\boxed{\bar a}} \underbrace{\left(\frac{z + \frac{b}{a}}{1 + \frac{\bar b}{\bar a} z} \right)}_{\psi_{-\frac{b}{a}}(z)}.$$

$$\psi_\alpha(z) = \frac{z - \alpha}{1 - \bar\alpha z} \qquad \alpha, a \to b. \qquad \frac{a}{\bar a} = e^{i\theta}$$

$$|\alpha| = \left|\frac{b}{a}\right| < 1.$$

Thm: $\mathrm{Aut}(\mathbb{D}) = PSU(1,1) = SU(1,1) / \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$

$\mathrm{Aut}(\mathbb{H})$? f:



$$C : \mathbb{H} \to \mathbb{D} : z \mapsto \frac{z - i}{z + i} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \circ z.$$

Cayley transform $\quad C^{-1} = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$

Obs: $\mathbb{D} \xrightarrow{C^{-1}} \mathbb{H} \xrightarrow{f} \mathbb{H} \xrightarrow{C} \mathbb{D}$.

$$\underbrace{\mathbb{D} \to \mathbb{D}}_{\in \operatorname{Aut}\mathbb{D}}$$

$\Rightarrow$ $C \circ f \circ C^{-1} \in PSU(1,1)$ $= \begin{pmatrix} a & b \\ \bar b & \bar a \end{pmatrix}$ $\Rightarrow$ $f = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ \bar b & \bar a \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$.

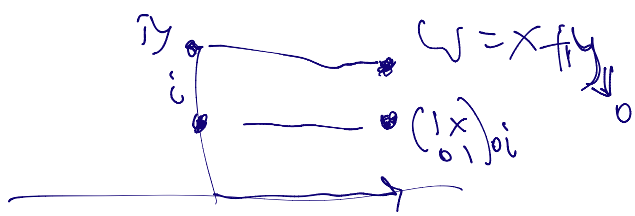
Exercise: work this out...

Claim: $PSL_2(\mathbb{R}) = \operatorname{Aut}(\mathbb{H})$

• $SL_2(\mathbb{R})$ acts transitively

Claim: $\forall w \in \mathbb{H}$, $\exists g \in SL_2(\mathbb{R})$ s.t. $g: i \mapsto w$.

$\forall z, w \in \mathbb{H}$, $\exists g \in SL_2(\mathbb{R})$: $g: z \mapsto w$.



$\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \circ z = y \cdot z$, $\in SL_2(\mathbb{R})$

$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : z \mapsto \dfrac{z+x}{0+1} = z+x$

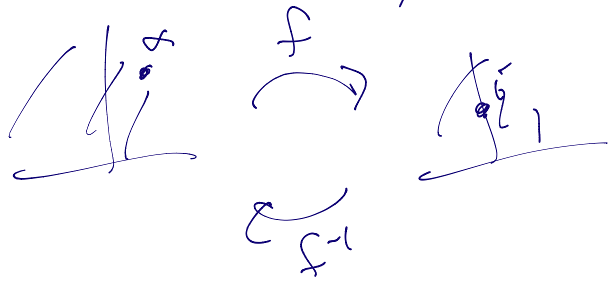
$\begin{pmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{pmatrix} : i \mapsto iy$.

$\underbrace{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{pmatrix}}_{g \in SL_2(\mathbb{R})} i = x+iy$.

$z \mapsto w$, $z \to i \to w$

$M_w = \begin{pmatrix} 1 & \operatorname{Re} w \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{\operatorname{Im} w} & 0 \\ 0 & 1/\sqrt{\operatorname{Im} w} \end{pmatrix} : i \mapsto w$.

If $f \in \underline{\mathrm{Aut}\,\mathbb{H}}$, let $\alpha := f^{-1}(i)$.



Let $g := f \circ m_\alpha$

$$g(i) = f(\underbrace{m_\alpha(i)}_{\alpha}) = i \qquad \left.\frac{z-i}{z+i}\right|_{z=i} = 0.$$

$h := \underbrace{C \circ g \circ C^{-1}}$ $\in \mathrm{Aut}\,\mathbb{D}$.

Then h ↺, $h(0)=0 \Rightarrow h(w) = \underline{e^{i\theta}} \cdot w$.

Exercise: $f \in PSL_2(\mathbb{R})$.

If $f : U \to \mathbb{D}$ $\underline{\text{Conf}}$ map. bij hol.

$\underline{\mathrm{Aut}\,U} \sim \underline{PSU(1,1)}$.

$$\{M_w\} = \Big\{\underbrace{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}}\underbrace{\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}}\Big\} = \Big\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\Big\} = B$$

(nilpotent)

Borel subgroup $= \mathbb{H}$.

$B\, i = \mathbb{H}$.

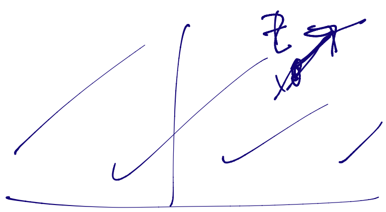
$G\, i = \mathbb{H}$ $\qquad \mathrm{Stab}_G\, i = K = SO(2)$.

sgn.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

$G = PSL_2(\mathbb{R}) \cong T^1\mathbb{H}$.



$G = B \cdot SO(2) = NAK$

"Iwasawa Coord."

Compact.

$\{f : \mathbb{H} \to \mathbb{D}\}$

$PGL_2^+(\mathbb{R}) \curvearrowright \mathbb{H}$. $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} z = \frac{\cancel{\lambda} az + \cancel{\lambda} b}{\cancel{\lambda} cz + \cancel{\lambda} d}$

$=$

$\|$ Component of $\quad GL_2(\mathbb{R}) = GL_2^+(\mathbb{R}) \sqcup GL_2^-(\mathbb{R})$

$PSL_2(\mathbb{R})$.

$J \in$

$$\begin{pmatrix}2&1\\1&1\end{pmatrix} \qquad PGL_2^+ \ni \begin{pmatrix}3&1\\1&1\end{pmatrix}\circ z = \frac{3z+1}{z+1}$$

$$\|$$

$$PSL_2 \ni \begin{pmatrix}\frac{3}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}\circ z = \frac{\frac{3}{\sqrt{2}}z+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}z+\frac{1}{\sqrt{2}}}.$$