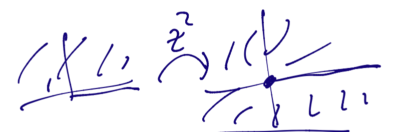
Recall: Def: $f : U \to V$ Conformal $\iff$ f holo, bijective & f^{-1} holo.

(U & V are conf if $\exists f$).

- f injective & holo $\Rightarrow$ $f' \neq 0$ • $\Rightarrow$ f^{-1} holo. automatic.

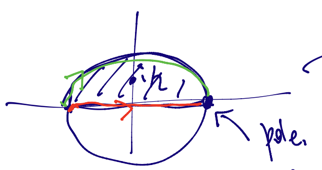
($\Rightarrow$ image open $V = \text{Image } f = f(U)$)

Ex 1: $\mathbb{H} \to \mathbb{D} : z \mapsto \frac{z-i}{z+i} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \circ z$ (frac linear transf).

Ex 2: $\mathbb{H} \to \mathbb{C} \setminus \{[0,\infty)\} : z \mapsto z^2$. 

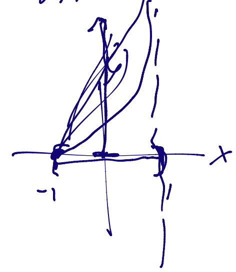
Ex 3: $f_3(z) = \frac{1+z}{1-z}$ on $\mathbb{D} \cap \mathbb{H}$ = upper half-disk

pole at $z=1$



$\to$? . Idea: Look at ∂U.

$f\big|_{[-1,1]} : x \mapsto \frac{1+x}{1-x}$ $\quad y = \frac{1+x}{1-x}$

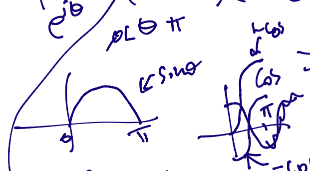


$f' = \frac{(1-z) + (1+z)}{(1-z)^2} \neq 0.$ $\quad y' \neq 0.$

$f : [-1,1] \to [0,\infty)$

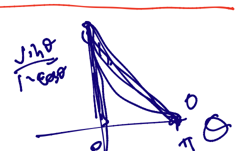
$f\big|_{e^{i\theta}} = \frac{(1+e^{i\theta})(1-e^{-i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})} = \frac{\cancel{1}+2i\sin\theta - \cancel{1}}{1-2\cos\theta+1}$

$0<\theta<\pi$



$= \boxed{\frac{(1)\sin\theta}{1-\cos\theta}} = i \cot\frac{\theta}{2}.$

$f\left(\frac{i}{2}\right) = \frac{\left(1+\frac{i}{2}\right)\left(1+\frac{i}{2}\right)}{\left(1-\frac{i}{2}\right)\left(1+\frac{i}{2}\right)}$



$\frac{1-\frac{1}{4}+i}{1+\frac{1}{4}}$

$\frac{(1+e^{i\theta})\, e^{-i\theta/2}}{(1-e^{i\theta})\, e^{-i\theta/2}} = \frac{e^{-i\theta/2} + e^{i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} = \frac{2\cos\frac{\theta}{2}}{-2i\sin\frac{\theta}{2}}$

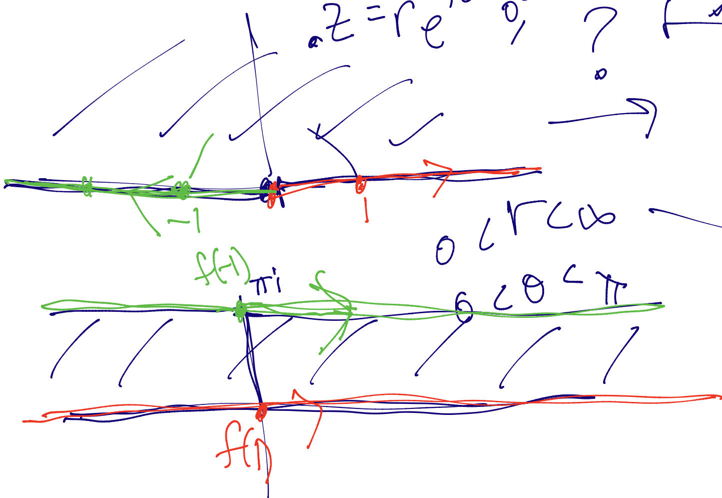
$= i \cot\frac{\theta}{2}.$

f_3:  , $\begin{pmatrix} 1 & 1 \\ -1 & +1 \end{pmatrix} \cdot z = f(z)$

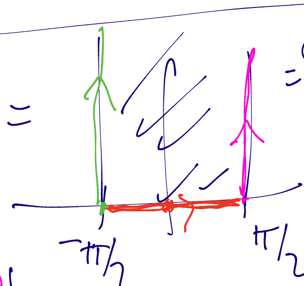
$f^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \circ z \mapsto \frac{z-1}{z+1}$ $\leftarrow$ well-def unless $z=-1$

Ex4: $f_4 = \log z$ (principal branch) $U = \mathbb{H}$.

$z = re^{i\theta}$, $0<\theta<\pi$? $\xleftrightarrow{\log}$ $\underbrace{\log r}_{(-\infty,\infty)} + \underbrace{i\theta}_{(0,\pi)}$.

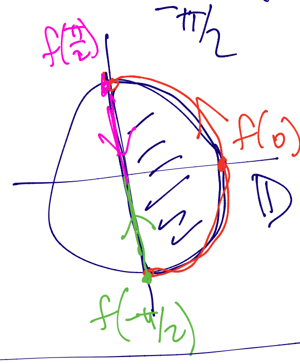


Ex5: $f_5 : z \mapsto e^{iz}$, $U =$



$= \{z = x+iy \mid |x| < \frac{\pi}{2}, y \geq 0\}$.

If $z = x+iy$, $e^{iz} = \underbrace{e^{ix}\cdot e^{-y}}_{1 \,\cdot\, |<1}$



Ex6: $f_6 : z \mapsto -\frac{1}{2}\left(z + \frac{1}{z}\right)$: $\mathbb{D}\cap\mathbb{H} \longrightarrow$?.

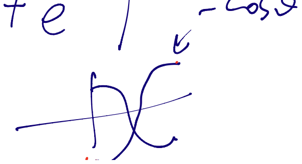
$\xrightarrow{?}$

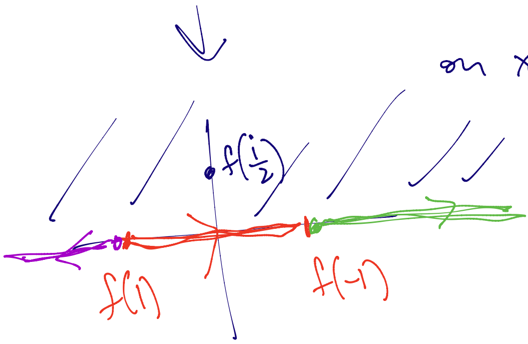
$\frac{z^2+1}{z}$ ← pole at $z=0$.

on $e^{i\theta}$, $0<\theta<\pi$

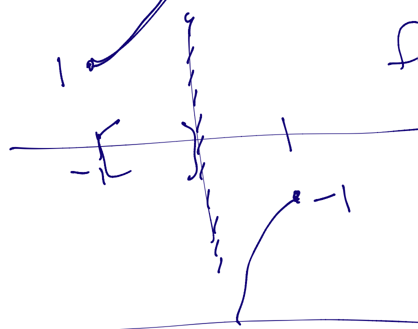
$f(e^{i\theta}) = -\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

$= -\cos\theta$





on $x\in[-1,1]$, $f(x) = \frac{-1}{2}\left(x+\frac{1}{x}\right)$.

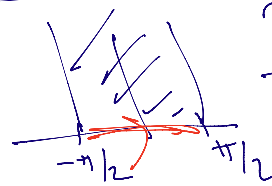


$f\left(\frac{i}{2}\right) = \frac{-1}{2}\left(\frac{i}{2}+\frac{2}{i}\right)$

$\frac{-1}{2}\left(\frac{i}{2}-2i\right)$

$-\frac{1}{2}\left(-\frac{3}{2}i\right)$

Ex 7. $f(z) = \sin z$ on

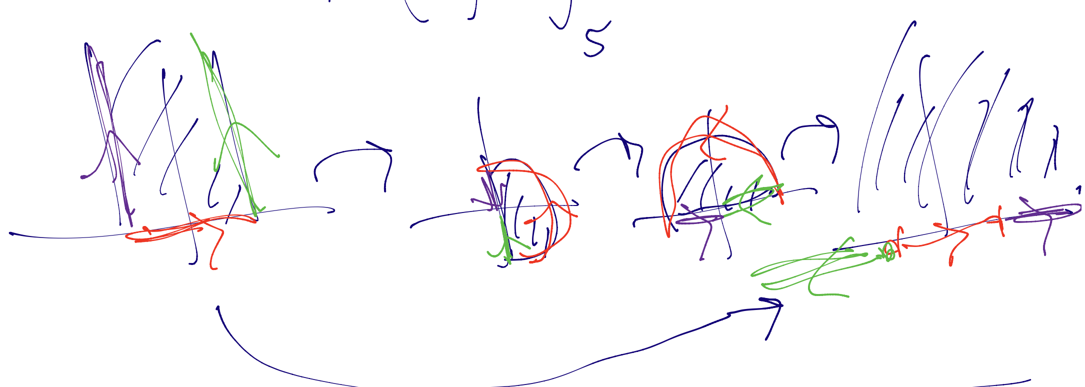


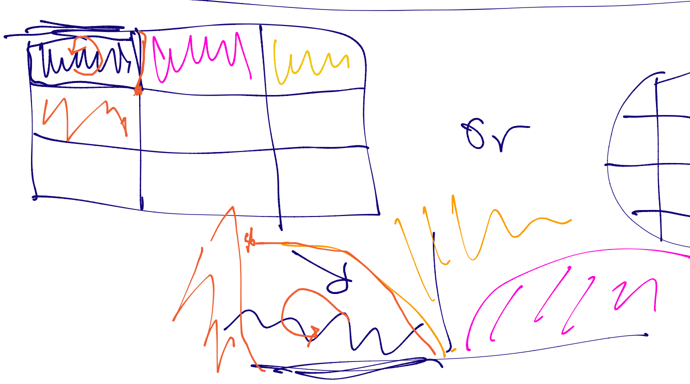
$\xrightarrow{???} \mathbb{H}$.

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i} = i\left(-\frac{1}{2}\right)\left(e^{iz}-\frac{1}{e^{iz}}\right) = -\frac{1}{2}\left(ie^{iz}+\frac{1}{ie^{iz}}\right)$$

$$\sin z = f(z) = f_6 \circ (i) \circ f_5$$

$\sin z$:





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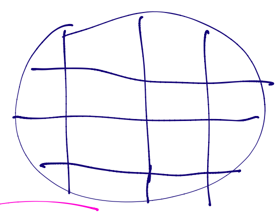

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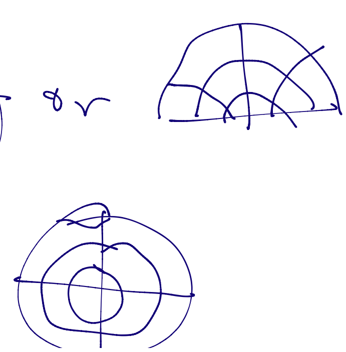
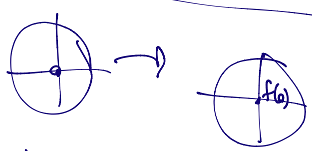


Obs: If $f: U \to \mathbb{D}$, are there other conformal maps? If f_1 is another, then:

$$f \circ f_1^{-1} : \mathbb{D} \to \mathbb{D}. \text{ So } f \circ f_1^{-1} \in \mathrm{Aut}(\mathbb{D})$$



"automorphism group" $=$ $\{f: \mathbb{D} \to \mathbb{D} \mid \text{bij, hol}\}$.

Schwarz Lemma: Let $f: \mathbb{D} \to \mathbb{D}$ hol (not nec. conformal) & assume $f(0) = 0$. Then: ($\Rightarrow |f(z)| \le 1$)

(i) $\forall z \in \mathbb{D}$, $|f(z)| \le |z|$. ✓

(ii) If $\exists\, z_0 \in \mathbb{D} \setminus \{0\}$ s.t. $|f(z_0)| = |z_0| \Rightarrow f$ is rotation, i.e. $f(z) = c \cdot z$, $|c| = 1$.

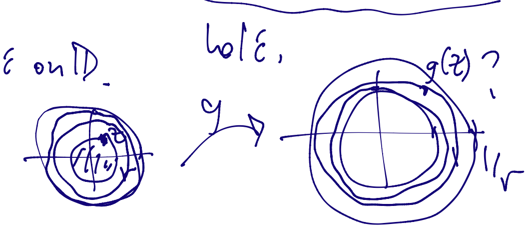
(iii) $|f'(0)| \le 1$. Moreover, if $|f'(0)| = 1 \Rightarrow f$ rotation.

Pf (i) Expand f about 0, $\frac{f(z)}{z} = \cancel{f(0)}^{0} + f'(0)\frac{z}{z} + a_2 \frac{z^2}{z} + \dots$

$g(z) := \frac{f(z)}{z}$ has removable sing at 0. $\Rightarrow$ hol on $\mathbb{D}$.

If $|z| = r$

$$|g(z)| = \left|\frac{f(z)}{z}\right| \le \frac{1}{r}.$$



So $\left| g(z)\big|_{\partial D_r(0)} \right| \le \frac{1}{r} \xRightarrow{\text{max mod}} \left| g(z)\big|_{D_r(0)} \right| \le \frac{1}{r}$.

Send $r \to 1$. $\Rightarrow |g(z)| \le 1$. $\Rightarrow |f(z)| \le |z|$ ✓.

Pf (ii): If $\exists\, z_0 \in \mathbb{D}\setminus\{0\}$ s.t. $|g(z_0)| = 1 \Rightarrow g$ constant!

$\Rightarrow f(z) = c \cdot z$. & $|g(z_0)| = |c| = 1$. $\Rightarrow f$ = rotation.

(iii): $g(z) = \frac{f(z)}{z} = \frac{f(z) - f(0)}{z - 0} \xrightarrow{z \to 0} f'(0) = g(0)$

& $|g(z)| \le 1 \Rightarrow |f'(0)| \le 1$. If $|f'(0)| = 1 = |g(0)|$.

$\xRightarrow{\text{max mod}}$ $g = \text{const.} \Rightarrow f$ is rotation.

Recall: "Blaschke factor" maps: $(|\alpha| < 1)$

$$\varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z} = \begin{pmatrix} -1 & \alpha \\ -\bar{\alpha} & 1 \end{pmatrix} \circ z$$

Claim: $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$ & hol. & bijective (i.e. $\varphi_\alpha \in \mathrm{Aut}\,\mathbb{D}$).

(Thm: $f \in \mathrm{Aut}\,\mathbb{D} \Rightarrow f(z) = e^{i\theta} \cdot \varphi_\alpha(z)$).

$\left(\mathrm{Aut}\,\mathbb{D} \cong \widehat{PSU(1,1)} \right)$

Pf of Claim: If ψ_α hol? Pole at $1-\bar{\alpha}z=0$,

$|z|=\left|\frac{1}{\bar{\alpha}}\right|>1$ pole is outside $\mathbb{D}$.

is it bijective? $\psi_\alpha = \psi_\alpha^{-1} = \begin{pmatrix}+1 & -\alpha \\ \bar{\alpha} & -1\end{pmatrix}(-1) = \begin{pmatrix}-1 & \alpha \\ -\bar{\alpha} & +1\end{pmatrix}$

So each ψ_α is an involution!

Exercise: $\psi_\alpha : \mathbb{D} \to \mathbb{D}$. ✓

Fact: $\psi_\alpha(0) = \alpha$, $\psi_\alpha(\alpha) = 0$.

If $f : U \to \mathbb{D}$ is conformal,

$\mathrm{Aut}\, U \ni \varphi$?

$f \circ \varphi \circ f^{-1} \in \mathrm{Aut}\, \mathbb{D}$.