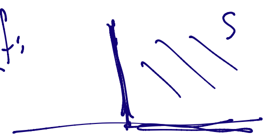
Last Time: Paley-Wiener Thm: $f|_{\mathbb{R}}$ has moderate decay. f entire, $|f(z)| < Ce^{2\pi M|z|}$,

$\Rightarrow \operatorname{supp} \hat{f} \subset [-M, M]$. $(\Leftarrow)$.

- Assume $\forall x,y$: $|f(x+iy)| < \dfrac{Ce^{2\pi M|y|}}{1+x^2}$ $\Rightarrow$ pulled contours (if $\xi > M$) $|\hat{f}(\xi)| < e^{-2\pi(\xi - M)y} \to 0$ for any y.
- Assume $|f(x+iy)| < Ce^{2\pi M|y|}$. Use $f_\varepsilon(z) = \dfrac{f(z)}{(1+i\varepsilon z)^2} \leftarrow \hat{f}_\varepsilon(\xi) = 0$ for $|\xi| > M$, $\hat{f}_\varepsilon \downarrow \hat{f}(\xi)$ as $\varepsilon \to 0$.
- Phragmen-Lindelöf:



If $|F|_{\mathbb{R}_+}| \le 1$ & $|F|_{i\mathbb{R}_+}| \le 1$ & $|F| \le Ce^{C|z|^{9/10}}$

$\Rightarrow |F| \le 1$ on all of S

Back to $\boxed{|f(z)| \le \overset{C=1}{C} e^{2\pi M|z|}}$ Let $F(z) = f(z)e^{2\pi i M z}$, $|F|_{\mathbb{R}_+}| \le 1$

$|f(x)| \le \frac{1}{1+x^2} \le 1$. On $i\mathbb{R}_+$, $|F(iy)| \le e^{2\pi M y} e^{-2\pi M y} = 1$.

$\Rightarrow |F(z)| = |f(z)| \cdot e^{-2\pi M|y|} \Rightarrow |f(z)| \le e^{2\pi M|y|}$.

§ Conformal Maps: Def: $f: U \to V$ is conformal if: $(U, V \subset \mathbb{C})$.
hol'c & bijective (1-1, & onto). (angle preserving)

Def: $U, V \subset \mathbb{C}$ are conformal if $\exists\, f: U \to V$ conformal.

Is $U \sim V$ conformal an equivalence relation? (Is f^{-1} hol'c???).

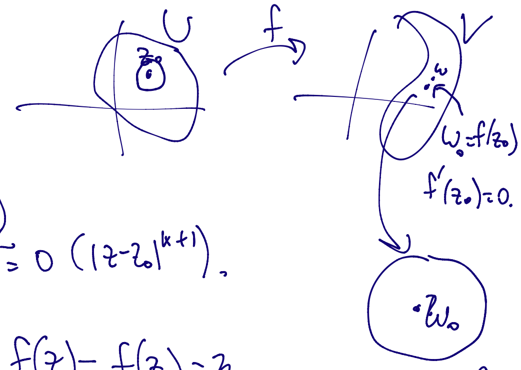
Lemma: Assume $f: U \to V$ hol'c & 1-1. Then $f'(z) \ne 0$ on U.

Aside: locally, $f(z) = f(z_0) + L(z - z_0) + o(|z - z_0|)$. ($\mathbb{R}^2$-diff'ble)

$f = u + iv$
$z = x + iy$

$L = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} = \cancel{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \pm 1.$

↳ pf: Assume $\exists z_0 \in U$: $f'(z_0)=0$. $(f \neq C)$.



Near z_0, $\boxed{f'(z)=0 \Rightarrow z=z_0}$

$$f(z) = f(z_0) + \cancel{f'(z_0)}(z-z_0) + \dots$$

$$= \underbrace{f(z_0)}_{= w_0} + \underbrace{a_k}_{\neq 0} (z-z_0)^k + \underbrace{g(z)}_{= O(|z-z_0|^{k+1})}, \quad k \geq 2$$

For w near w_0, $w = w_0 + \eta$ (η small), look at,

$$f(z) - w = f(z) - f(z_0) - \eta$$

$$= \underbrace{a_k (z-z_0)^k - \eta}_{F(z)} + \underbrace{g(z)}_{\to 0 \text{ as } z \to z_0}$$

On $\partial D_\varepsilon(z_0)$, $|F(z)| > |g(z)|$.

$\Rightarrow$ (Rouché): F & $F+g$ have same # zeros in $D_\varepsilon(z_0)$. $F(z)=0$ has $k \geq 2$ roots.

$\Rightarrow$ $f(z)-w$ has at least 2 roots. one is z_1, z_2.

Are they distinct? If not, then $z_1 = z_2$, $f'(z_1)=0$. But $f'(z_1) \neq 0$. ※.

Near z_1, $f(z) = \underbrace{f(z_1)}_{w} + \underbrace{f'(z_1)(z-z_1)}_{=0} + \dots$ Locally, $f(z) - w = O((z-z_1)^2)$, as $z \to z_1$.

Lemma: If $f: U \to V$ is conformal

$\Rightarrow f^{-1}$ is hol'c.

Pf. Let $g = f^{-1}$. Look at $\dfrac{g(w) - g(w_0)}{w - w_0} = \dfrac{1}{\frac{w-w_0}{g(w)-g(w_0)}} = \dfrac{1}{\frac{f(z)-f(z_0)}{z-z_0}} \to \dfrac{1}{f'(z_0)} \neq 0$.

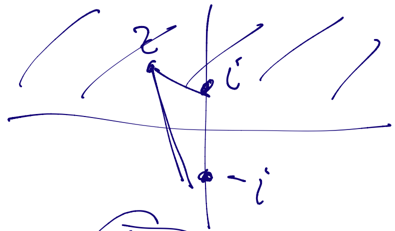
$w_0 = f(z_0)$, $w = f(z)$ ($\lim = g'(w_0)$)

$$\boxed{f(g(z)) = z \Rightarrow f'(g(z)) \cdot g'(z) = 1.}$$

$$g'(w_0) = \frac{1}{2\pi i} \int_{\partial D(w_0)} \frac{g(w)}{(w-w_0)^2} \, dw$$

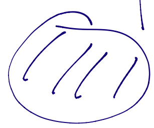
$\| \; g'(f(z_0))$

$$\Rightarrow g'(z) = \frac{1}{f'(g(z))} \qquad \boxed{g'(w_0) = \frac{1}{f'(g(w_0))}}$$

Ex: 

$\mathbb{H} = \{z = x+iy \mid y>0\}$.

non bdd

bdd 

$\mathbb{D} = \{z \mid |z|<1\}$.

$f: z \in \mathbb{H} \to \frac{z-i}{z+i} \in ?$. Is $\left|\frac{z-i}{z+i}\right| < 1$ for $z \in \mathbb{H}$? $\Leftarrow \mathbb{D}$.

Dist to i < Dist to $-i$ $\Rightarrow$ $|z-i| < |z+i|$.

$\Updownarrow$

$\mathbb{H}$

"fractional linear"

Is f hol?. $z \neq -i$ ✓,

If $w = \frac{z-i}{z+i}$, $z = ?$. $z = \frac{iw+i}{-w+1} = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \circ w$

$wz + iw = z - i$

Exercise: $(g_1 g_2) \circ z = g_1 \circ (g_2 \circ z)$

Give an action of $\cancel{M_{2\times2}(\mathbb{C})}$ $PGL_2(\mathbb{C})$ on $\mathbb{C}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \circ z = \frac{1\cdot z + 0}{0 \cdot z + 1} = z$. $\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \circ z := \frac{\lambda a z + \lambda b}{\lambda c z + \lambda d}$.

$\Rightarrow (g^{-1} \cdot g) \circ z = z = g^{-1} \circ (g \circ z)$ If $f(z) = \frac{z-i}{z+i} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \circ z$

$f^{-1}(z) = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$

$PGL_2(\mathbb{C}) = GL_2(\mathbb{C}) / \mathbb{C}^\times = Z$ = center.

Really: $GL_2(\mathbb{C}) \circlearrowright \mathbb{C}^2 \rightsquigarrow \underbrace{PGL_2(\mathbb{C})}\circlearrowright \underbrace{\mathbb{P}^1\mathbb{C}}$

$\mathbb{P}^1\mathbb{C} = \mathbb{C}^2\setminus\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\} \Big/ \mathbb{C}^\times$.

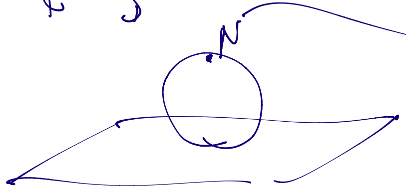
$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\circ\begin{pmatrix}z\\w\end{pmatrix}=\begin{pmatrix}az+bw\\cz+dw\end{pmatrix}$$

$$\mathbb{P}^1\mathbb{C} \ni \begin{pmatrix}z\\w\end{pmatrix} \sim \begin{pmatrix}\lambda z\\ \lambda w\end{pmatrix}\quad \forall \lambda\in\mathbb{C}^\times.$$

$\hat{\mathbb{C}} \cong S$ projective complex line:

$\begin{pmatrix}z\\w\end{pmatrix}$: $w\neq 0$: $\begin{pmatrix}z\\w\end{pmatrix} \sim \begin{pmatrix}zw^{-1}\\1\end{pmatrix}$, $zw^{-1}\in\mathbb{C}$

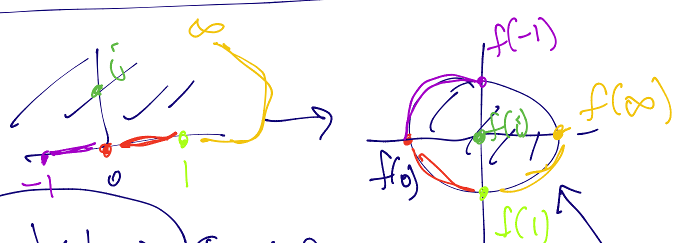
$\begin{pmatrix}z\neq 0\\0\end{pmatrix}\sim\begin{pmatrix}1\\0\end{pmatrix}$ ← pt at ∞



Say $w\neq 0 \rightsquigarrow w=1$: $\begin{pmatrix}az+b\cdot 1\\cz+d\cdot 1\end{pmatrix}\sim\begin{pmatrix}\frac{az+b}{cz+d}\\1\end{pmatrix}$.

$f: z\mapsto \dfrac{z-i}{z+i}$

$f^{-1}: w\mapsto \dfrac{iw+i}{-w+1}$ ← $|w|<1 \Rightarrow \mathrm{Im}>0$.



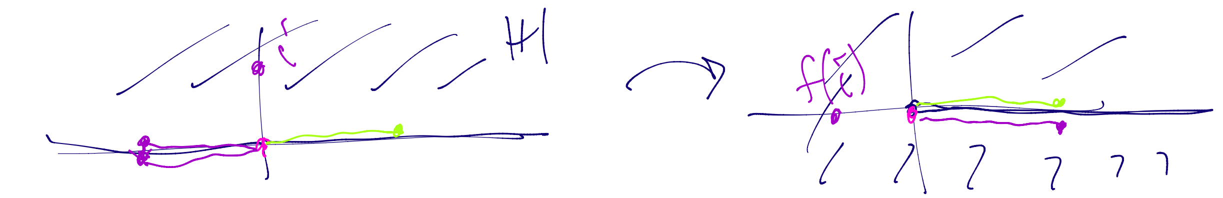
Good way to analyze these: look at γ's?.

$$f\big|_{\gamma=\mathbb{R}}: x\mapsto \frac{(x-i)(x-i)}{(x+i)(x-i)} = \frac{x^2-1-2xi}{x^2+1}\in\partial\mathbb{D}.$$

Algebra: $\left(\dfrac{x^2-1}{x^2+1}\right)^2+\left(\dfrac{-2x}{x^2+1}\right)^2=1$.

Ex: $f(z)=z^2$. If $z=re^{i\theta}$, $z^2=r^2e^{2i\theta}$.

$0<\theta<\pi \Rightarrow 0<2\theta<2\pi$



$g(D)$ $\quad z \in U \xrightarrow{f} w \in V$ $\quad w_0$ $\quad D$

z_0

$$g'(w_0) = \frac{1}{2\pi i}\int_{\partial D} \frac{g(w)}{(w-w_0)^2}\,dw = \boxed{\frac{1}{2\pi i}\int \frac{z}{(f(z)-f(z_0))^2}\, f'(z)\,dz}$$

$\frac{1}{f'(z_0)}$

$\partial g(D) \ni z_0$

$\| z_0 ?$

$\frac{1}{f'(g(w_0))}$ $\quad z_0$

$w = f(z)$

$$f(z) = f(z_0) + f'(z_0)(z-z_0)\left[1 + \frac{f''(z_0)}{f'(z_0)}(z-z_0) + \dots\right]$$

$$\approx \frac{1}{2\pi i}\int_{\partial g(D)} \frac{z\, f'(z)}{f'(z_0)^2 (z-z_0)^2 (---)}\,dz.$$

$$\mathrm{Res}_{z=z_0} \frac{z\cdot f'(z)}{(z-z_0)^2}$$

$$= \left(\frac{\partial}{\partial z}\left(\frac{z\cdot f'(z)}{\cancel{(z-z_0)^2}}\right)\cancel{(z-z_0)^2}\right)_{z\to z_0}$$

$$1\cdot f'(z) + z\cdot f''(z)\Big|_{z_0}.$$

$$\mathrm{Res}_{z=z_0} \frac{z f'(z)}{(f(z)-f(z_0))^2} = \frac{\partial}{\partial z}\left(\frac{z\cdot f'(z)\,(z-z_0)^2}{(f(z)-f(z_0))^2}\right)\Bigg|_{z\to z_0} \Rightarrow \frac{1}{f'(z_0)}$$

$$\int_{\partial T} g(w)\, dw = \oint_{\partial g(T)} z f'(z)\, dz = 0.$$

$w = f(z)$