Recall: $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx \quad \left(= \langle f, e_\xi \rangle \leftarrow \text{continuous spectrum} \right)$

Thm: $f \in \mathfrak{F}_a$ (holo, moderate decay in x, uniform in $|y|<a$). $|f(x+iy)| \le \frac{C}{1+x^2}$

$\Rightarrow \forall b<a, \ |\hat{f}(\xi)| \le C e^{-2\pi b|\xi|}$

Thm: f of moderate decay on $\mathbb{R}$: Suppose $\hat{f}$ exists & $|\hat{f}(\xi)| \le C \cdot e^{-2\pi a |\xi|}$. Then f has holo extension to $|\mathrm{Im}\, z| < a$. I.e. $\exists\, g : \{|\mathrm{Im}\, z|<a\} \to \mathbb{C}$ holo & $g|_{\mathbb{R}} \equiv f$.

Pf: Let $g_R(z) := \int_{-R}^{R} \underbrace{\hat{f}(\xi) e^{2\pi i z \xi}}_{F(z,\xi)} d\xi$. $\leftarrow$ Claim: holo. ✓

Aside: Thm: If $F : (z, \xi) \in \Omega \times [a,b] \to \mathbb{C}$ such that:

- $\forall$ fixed ξ, $F(\cdot, \xi)$ is holo $: \Omega \to \mathbb{C}$,
- jointly cont in (z, ξ).

Then $G(z) := \int_a^b F(z,\xi)\, d\xi : \Omega \to \mathbb{C}$ is holo ✓

Recall Morera: (Converse Goursat: f holo, $\int_{\partial T} f = 0$) If f cont $\forall T \subset \Omega$, $\int_{\partial T} f = 0 \Rightarrow f$ holo. (only care about F_ξ)

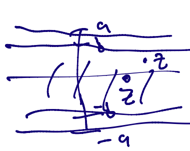
($\Rightarrow$ $f_n \to f$ unif on compacta $\Rightarrow f$ is holo.) (f_n holo)

Take $T \subset \Omega$, ∂T: $\int G(z)\, dz = \underbrace{\int_{\partial T} \int_a^b F(z,\xi)\, d\xi\, dz}_{\text{Converges } F \to |F|} \underset{\text{Fubini-Tonelli}}{=} \int_a^b \underbrace{\int_{\partial T} F(z,\xi)\, dz}_{\text{Goursat} \Rightarrow 0}\, d\xi = 0$, ($G$ cont; For ξ fixed, holo in z)

$\Rightarrow G =$ holo.

$\Rightarrow g_R(z) := \int_{-R}^{R} \hat{f}(\xi) e^{2\pi i z \xi} d\xi$ is holo on $\Omega = \mathbb{C}$.

Fix $0<b<a$. Fix z with $|\mathrm{Im}(z)| \le b$. $\quad \left(|\hat{f}(\xi)| \le C e^{-2\pi a|\xi|} \right)$



Then $|g_R(z)| \le \int_{-R}^{R} C \cdot \underbrace{e^{-2\pi a|\xi|}}_{\text{has more decay}} \cdot \underbrace{\left| e^{2\pi i z \xi} \right|}_{\le e^{2\pi b|\xi|} \text{ has growth}} d\xi \le C$; $\quad e^{-2\pi|\xi|(a-b)}$; has more decay than $\to$ has growth

well-def. $\boxed{g(z) := \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i z \xi} d\xi}$

Look at $|g_R(z) - g(z)| \le \int_R^{\infty} C\cdot e^{-2\pi|\xi|(a-b)}\,d\xi \le \underbrace{C\cdot e^{-2\pi R(a-b)} \to 0}_{\text{uniform in } z,}$

$g_R(z) \to g(z)$ unif on compacta $\subset \{|\operatorname{Im} z| < b\}$.

$\Rightarrow g(z)$ hol'c on $|\operatorname{Im} z| \le b$, $\forall\, b<a \Rightarrow g(z)$ holc on $|\operatorname{Im} z|<a$.

Cor: (baby Heisenberg): If $|\hat f(\xi)| \le Ce^{-2\pi a|\xi|}$ & f has compact support $\operatorname{supp} f \subset [-M, M]$

$\Rightarrow f = 0$.

Obs: Solving ODEs becomes solving polynomials under $\wedge$:

$\widehat{Df=0}$ $\quad$ $P(\xi)\cdot\hat f = 0$.

$f'' + f' - f = 0 \qquad \widehat{f'}(\xi) = \int_{\mathbb{R}} f'(x)\, e^{-2\pi i x\xi}\,dx$

$\widehat{f'} = c\cdot\xi\cdot\hat f \qquad \boxed{\widehat{\partial_x f}(\xi) = c\,\xi\cdot\hat f} \qquad = 2\pi i\xi \int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi}\,dx$

Thm (Paley-Weiner): Assume f is cont on $\mathbb{R}$ & of moderate decrease. Then

$\operatorname{supp}\hat f \subset [-M, M] \iff f$ is extendable to entire & $|f(z)| \le C\cdot e^{2\pi M|z|}$ (Same M!)

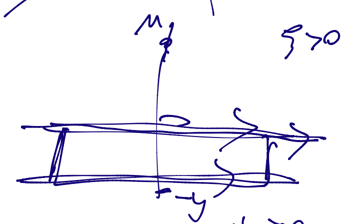
Pf $\Rightarrow$) If $\operatorname{supp}\hat f \subset [-M, M]$, $g(z) := \int_{-M}^{M} \hat f(\xi)\, e^{2\pi i z\xi}\,d\xi$ $\leftarrow$ holc & entire & $g|_{\mathbb{R}} = f$.

$= \int_{-\infty}^{\infty}$.

$|g(z)| \le \underbrace{\int_{-M}^{M} C\cdot e^{2\pi M|z|}\,d\xi}$. ✓.

Pf $\Leftarrow$): Have: f is entire, $|f(z)| \le C\cdot e^{2\pi M|z|}$. Claim: $\forall |\xi| > M$, $\hat f(\xi) = 0$.

Lemma 1: If f is entire, $|f(x+iy)| \le \dfrac{Ce^{2\pi M|y|}}{1+x^2}$



Pf: Look at $\hat f(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi}\,dx$

Assume $\xi > M$.

$|\hat f(\xi)| \le \int_{\mathbb{R}} |f(x-iy)\, e^{-2\pi i\xi(x-iy)}|\,dx$

$\le \int_{\mathbb{R}} \dfrac{C\cdot e^{2\pi M|y|}}{1+x^2}\, e^{-2\pi\xi y}\,dx \le C\cdot e^{2\pi y\overbrace{(M-\xi)}^{<0}} \to 0$, true, $\forall y$. $y\to\infty$

indep of y.

Similar for $\xi < -M$, $x-iy \mapsto x+iy$...

Lemma 2: If f is entire & $|f(x+iy)| \le Ce^{2\pi M|y|}$. Then $|\xi| > M \Rightarrow \hat f(\xi) = 0$.

(& $f|_{\mathbb{R}}$ has moderate decrease)

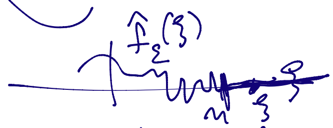
Pf: Consider: $(\varepsilon>0)$ $f_\varepsilon(z) := \dfrac{f(z)}{(1+i\varepsilon z)^2}$. Say $z = x-iy$

$$\left| f_\varepsilon(x+iy) \right| \le \frac{C\cdot e^{2\pi M|y|}}{1+\varepsilon^2 x^2}$$

Lemma 1 applies

$$|1+i\varepsilon z|^2 = |1+i\varepsilon x+\varepsilon y|^2 = (1+\varepsilon y)^2+\varepsilon^2x^2 \ge 1+\varepsilon^2x^2 \ge 1.$$

$\xi > M \Rightarrow \hat f_\varepsilon(\xi) = 0$



Look at $\left| \hat f_\varepsilon(\xi) - \hat f(\xi) \right| \le \int_{\mathbb{R}} \left| \dfrac{1}{(1+i\varepsilon x)^2} - 1 \right| |f(x)| \left| e^{-2\pi i x\xi} \right| dx$

$|f(x)| \le \frac{1}{1+x^2}$; $\left|\frac{1}{(1+i\varepsilon x)^2} - 1\right|$ bdd, $\to 0$ as $\varepsilon \to 0$; $\le C$ indep of ξ, $\to 0$ as $\varepsilon\to 0$.

0, $\forall \xi > M$, $\forall \varepsilon > 0$. indep of ε.

$\Rightarrow \hat f(\xi) = 0$.

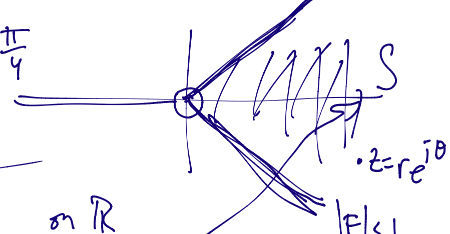
Thm (Phragmen-Lindelof): Assume $|f(x)| \le 1$ $\quad f|_{\mathbb{R}}| \le 1$. $|f|\le 1$

& $\forall z\in\mathbb{C}$, $|f(z)| \le e^{2\pi M|z|}$. Then $|f(x+iy)| \le e^{2\pi M|y|}$. ($\Rightarrow$ Paley-Wiener)

Thm Version 2: Suppose F hol'c inside $S = \{ |\arg z| < \frac{\pi}{4} \}$ $|F|\le 1$

& Cont on $\bar S$. Assume $|F| \le 1$ on $|\arg z| = \frac{\pi}{4}$

& $|F(z)| \le e^{2\pi M|z|} \Rightarrow |F| \le 1$ on S.



(Enemy: $f(z) = e^{z^2}$ if $z = te^{i\pi/4} \Rightarrow z^2 = \pm it^2$. on $\mathbb{R}$
$f(z) = e^{\pm it^2}$ $|f| = 1$. $f \to \infty$.

Pf. Idea: Kill enemy: Set $F_\varepsilon(z) := F(z) e^{-\varepsilon z^{3/2}}$ ← principal branch of log.

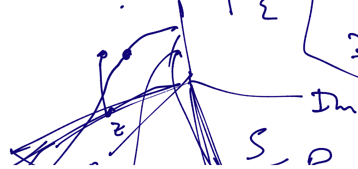
$$\left| e^{-\varepsilon z^{3/2}} \right| = e^{-\varepsilon r^{3/2} \cos\frac{3\theta}{2}} \qquad (3/2) \to 2X \qquad \cos\frac{3\theta}{2} \ge 0.01.$$

$\left|\frac{3\theta}{2}\right| \le \frac{3\pi}{8} < \frac{\pi}{2}$ $\qquad (re^{i\theta})^{3/2} = r^{3/2} e^{\frac{3}{2}i\theta}$ $\qquad z^{3/2} = \exp\left(\frac{3}{2}\log z\right)$.

$|F_\varepsilon(r)| \le e^{2\pi M|z|} e^{-\varepsilon|z|^{3/2}\cdot 0.01} \to 0$ as $|z|\to\infty$.

F_ε General: $\frac{3}{2} < 2$.

Let $L = (\sup_{z\in S} |F_\varepsilon(z)| \in \bar S \cap \{|z| < R\}$



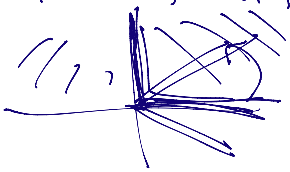
$\exists\, z \in \bar{S}$ s.t. $|F_\varepsilon(z)| = L$. Can $z \in S$? No! (max modulus).

$\Rightarrow z \in \partial \bar{S}$. $\Rightarrow L = |F_\varepsilon(z)| \leq 1 \Rightarrow \forall\, z \in \bar{S},\ |F_\varepsilon(z)| \leq 1$.

Send $\varepsilon \to 0$, $|F(z)| \leq 1$, on all of S.

Shown: $|F|_{\partial S} \leq 1$ & $|F| \leq e^{2\pi M \cdot |z|^{2-\varepsilon}} \Rightarrow |F| \leq 1$ on all of S.

For f entire, consider $F(z) = f(z)\, e^{2\pi i M z}$



$f(z) \mapsto f(e^{\pi i/4} z)$ on $\mathbb{R}_+$, $|F| = |f(x)| \leq 1$.

on $i\mathbb{R}_+$, $|F| \leq \underbrace{|f(iy)|}_{\leq e^{2\pi M y}} \cdot e^{-2\pi M y} \leq 1$.

$\Rightarrow |F| \leq 1$ on $\llcorner$

$\underbrace{|f(z)\, e^{2\pi i M z}|}_{} \Rightarrow |f| \cdot e^{-2\pi M y} \leq 1. \Rightarrow \underline{|f(x+iy)| \leq e^{2\pi M y}}$.

Recap: $|f(z)| \leq C \cdot e^{2\pi M |z|} \Rightarrow |f(z)| \leq C \cdot e^{2\pi M |y|}$.