Last time: • $\mathcal{F}_a = \{ f$ hol. on $|\text{Im} z| < a$ 
& $\exists C > 0 : \forall |y| < a, \forall x \in \mathbb{R}$, $|f(x+iy)| \le \frac{C}{1+x^2}$ "moderate decay" $\}$.

• $\mathcal{F} = \bigcup_{a>0} \mathcal{F}_a$.

• Thm: $f \in \mathcal{F}_a \Rightarrow \forall b<a, |\hat{f}(\xi)| \le C e^{-2\pi b|\xi|}$ (pf p'll contours) $\leftarrow$ exp decay $\vee$ super-poly. ($\xi \to \infty$).

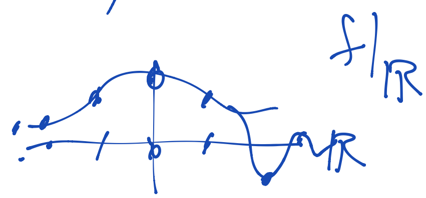
• Thm: $f \in \mathcal{F}_a \Rightarrow$ F. Inversion: $\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx$.

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

Thm (Poisson Summation): $f \in \mathcal{F}_a$, then

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} \hat{f}(m)$$

$n \in \mathbb{Z}$ ← geometric; $m \in \mathbb{Z}$ ← spectrum



First hint "duality" (Baby Trace Formula)

Pf 1: $f \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Define $\boxed{F(x) := \sum_{n \in \mathbb{Z}} f(x+n)}$.

F is well-defined on $\mathbb{R}/\mathbb{Z}$ because $F(x+1) = F(x)$.

$\Rightarrow$ By "baby Fourier", $F(x) = \sum_{m \in \mathbb{Z}} \hat{F}(m) e_m(x)$ $\quad e_m(x) = e^{2\pi i m x}$.

where $\hat{F}(m) = \langle F(\cdot), e_m(\cdot) \rangle$. $\quad \Delta \circlearrowright L^2(\mathbb{R}/\mathbb{Z}, dx) = \mathcal{H} = \bigoplus_{m \in \mathbb{Z}} \mathbb{C} e_m$ ← orthonormal basis

$\Delta = \partial_{xx}$ $\quad \langle f, g \rangle = \int_{\mathbb{R}/\mathbb{Z}} f(x) \overline{g(x)} dx$

What are eigenfunctions of Δ on $\mathbb{R}/\mathbb{Z}$.

$\partial_{xx} f = \lambda f \Rightarrow f = c_1 e_m + c_2 e_{-m}$ $\quad \Delta e_m = \underbrace{(-4\pi^2 m^2)}_{\le 0} e_m$ (diagonalizes Δ.)

"sesquilinear": linear, conj-linear in g.

Δ is a self-adjoint, neg def operator. $\langle \Delta f, g\rangle = \int_{\mathbb{R}/\mathbb{Z}} \partial_{xx} f\, \bar{g} = -\int_{\mathbb{R}/\mathbb{Z}} \partial_x f\, \partial_x \bar{g}\, dx = +\int_{\mathbb{R}/\mathbb{Z}} f\, \Delta\bar{g}\, dx = \langle f, \Delta g\rangle.$ Δ^*

$\langle \Delta f, f\rangle = -\int \partial_x f \cdot \overline{\partial_x f} = -\int |\partial_x f|^2 \leq 0.$ $\quad \operatorname{Spec}\Delta = \left\{ -4\pi^2 m^2 : m \in \mathbb{Z} \right\}.$

If f eigenfunct of Δ, $\Delta f = \lambda f$, $\langle \Delta f, f\rangle = \lambda \cdot \|f\|^2$ ≥ 0. ?

$(\Leftarrow)$ $(\Rightarrow)$

$\hat{F}(m) = \int_{\mathbb{R}/\mathbb{Z}} F(x)\, \overline{e_m(x)}\, dx$

$\| \int_{D}$ $\leftarrow \mathbb{R}/\mathbb{Z}$ $\leftarrow$ indep of choice of fund dom.

$\leftarrow$ choose some fixed domain.

$$\int_D \sum_{n\in\mathbb{Z}} f(x+n)\, \overline{e_m(x)}\, dx \overset{\checkmark}{=} \sum_{n\in\mathbb{Z}} \int_{D \neq \mathbb{R}/\mathbb{Z}} f(x+n)\, \overline{e_m(x)}\, dx.$$ "unfolding"

$x \mapsto x - n$

$$= \sum_{n\in\mathbb{Z}} \int_{D+n} f(x)\, \underbrace{\overline{e_m(x-\cancel{n})}}_{\text{no } n\text{'s}}\, dx = \int_{\mathbb{R}} f(x)\, e^{-2\pi i m x}\, dx = \hat{f}(m)$$

so:

$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n) = \sum_{m\in\mathbb{Z}} \hat{F}(m)\, e_m(x) = \sum_{m\in\mathbb{Z}} \hat{f}(m)\, e_m(x).$$ $= \hat{F}(m)$

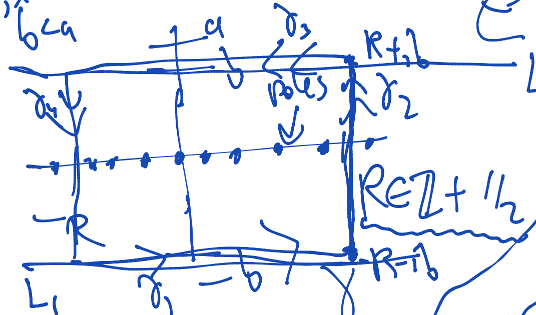
$$F(0) = \sum_{n\in\mathbb{Z}} f(n+0) = \sum_{m\in\mathbb{Z}} \hat{f}(m) \cdot 1 \quad \checkmark$$

Pf 2: (Complex Analysis) Want: $\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m).$

Look at: $g(z) = \dfrac{f(z)}{e^{2\pi i z} - 1}$ ← holo on $|\mathrm{Im}\, z| < a$.

← has simple poles at $z \in \mathbb{Z}$.

Fix $b < a$



$$\int_\gamma g(z)\,dz = 2\pi i \sum_{|n|<R} \operatorname{Res}_{z=n} g$$

Need: $\boxed{\operatorname{Res}_{z=n} \dfrac{f(z)}{e^{2\pi i z}-1} = \dfrac{f(n)}{2\pi i}}$

$= \lim_{z\to n} \dfrac{f(z)}{e^{2\pi i z}-1}(z-n) \rightsquigarrow \dfrac{e^{2\pi i z} - \overset{1}{\overset{\shortparallel}{e^{2\pi i n}}}}{z-n} \to \dfrac{d}{dz} e^{2\pi i z}\Big|_{z=n} = e^{2\pi i z}\cdot(2\pi i)\Big|_{z=n}$

$$\int_\gamma g = 2\pi i \sum_{|n|<R} \frac{f(n)}{2\pi i} \to \sum_{n\in\mathbb{Z}} f(n)$$

$$= \int_{\gamma_1} + \int_{\gamma_2} - \int_{\gamma_3} + \int_{\gamma_4}$$

($\int_{\gamma_2} \to 0$, $\int_{\gamma_4} \to 0$)

$$\left|\int_{\gamma_2}\right| \le \int_{-b}^{b} \left|\frac{f(R+iy)}{e^{2\pi i (R+iy)}-1}\cdot i\right| dy$$

$\gamma_2 \leftarrow z = R+iy,\ -b<y<b$

$$\le \frac{K}{1+R^2}\cdot \int_{-b}^{b} \frac{1}{\left|-\underbrace{e^{-2\pi y}}_{>0} - 1\right|}\,dy$$

$\left|-e^{-2\pi y} - 1\right| = 1 + e^{-2\pi y} > 1$

$\to 0$ as $R\to\infty$.

$$\int_\gamma g \to \int_{L_1} \frac{f(z)}{\underbrace{e^{2\pi i z}}_{|\cdot|>1}-1}\,dz + \int_{L_2} \frac{f(z)}{\underbrace{-e^{2\pi i z}}_{|\cdot|<1}+1}\,dz$$

$\downarrow$

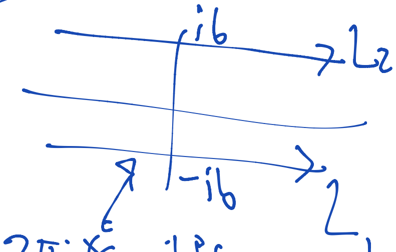
$\sum_n f(n)$

Recall: $|\lambda|<1 \Rightarrow \sum_{m\ge 0} \lambda^m = \dfrac{1}{1-\lambda}$.

If $|\lambda|>1 \Rightarrow \lambda^{-1}\sum_{m\ge0} \lambda^{-m} = \dfrac{1}{1-\frac{1}{\lambda}} = \dfrac{\lambda}{\lambda-1}$, $\quad \dfrac{\lambda-1}{\lambda}$

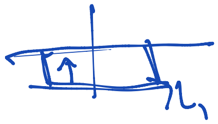
$$\int_{L_2} f(z) \sum_{m\ge0} \underbrace{e^{2\pi i m z}}_{|\cdot| = e^{-2\pi m b}}\,dz \quad \text{abs conv}$$

$$\int_{L_1} f(z)\, e^{-2\pi i z} \sum_{m\ge0} e^{-2\pi i m z}\,dz$$



$$\left|e^{2\pi i (x - ib)}\right| = e^{+2\pi b} > 1.$$

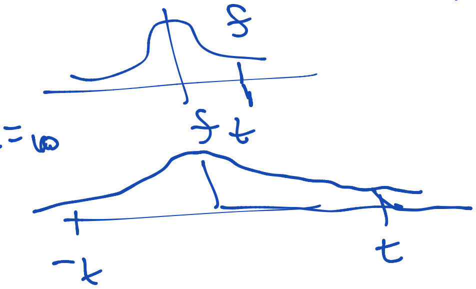
$$\int_\gamma g \to \int_{L_1} + \int_{L_2} = \sum_{m\ge0} \int_{L_1} f(z)\, e^{-2\pi i (m+1) z}\,dz + \sum_{m\ge0} \int_{L_2\to\mathbb{R}} f(z)\, e^{2\pi i m z}\,dz$$

 $= \sum_{m\geq 0}\hat{f}(m+1) + \sum_{m\geq 0}\hat{f}(-m) = \boxed{\sum_{m\in\mathbb{Z}}\hat{f}(m)}$

"$\boxed{\sum_{n\in\mathbb{Z}} f(n)}$"

Application: ("$\Longleftarrow$") Functional Equation for Riemann Zeta Function

Suppose $f_t(x) = f\left(\frac{x}{t}\right)$, $t\to\infty$



$$\hat{f}_t(\xi) = \int_{\mathbb{R}} \underbrace{f_t(x)}_{f\left(\frac{x}{t}\right)} e^{-2\pi i x\xi}\,dx \overset{x\mapsto xt}{=} t\underbrace{\int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi t}\,dx}_{\hat{f}(\xi t)}.$$

Poisson Summation:

$$\sum_{n\in\mathbb{Z}} f_t(n) = \boxed{\sum_n f\left(\frac{n}{t}\right) = t\sum_m \hat{f}(mt)} = \sum_m \hat{f}_t(m)$$

E.g., $f(x) = e^{-\cancel{2}\pi x^2}$, $\hat{f}(\xi) = e^{-\cancel{2}\pi\xi^2}$,

Say $t = 100$

terms ≈ 100

terms $= 1$ $(m=0)$.

$$\sum_n e^{-2\pi n^2/t^2} \approx t$$

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f[x_] := E^(-Pi x^2); (* Not E^(-2 Pi x^2) !... *)
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In[2]:= Plot[f[x], {x, -2, 2}]
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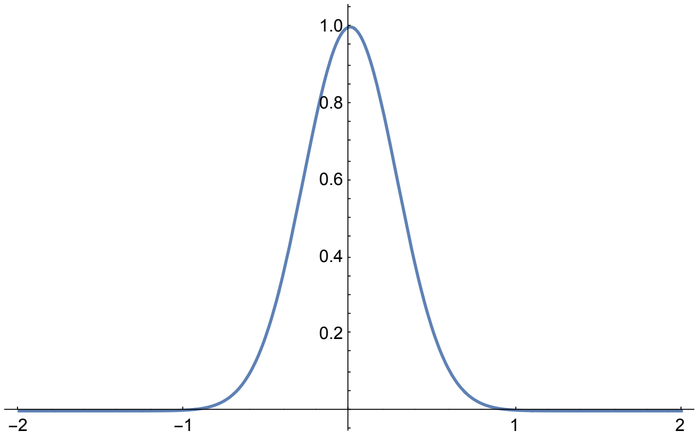
Out[2]=




```
In[17]:= partialSum[n_, t_] := Sum[f[m / t], {m, -n, n}];

In[18]:= partialSum[200, 100] // N

Out[18]= 99.9999

In[19]:= partialSum2[n_, t_] := t Sum[f[m t], {m, -n, n}];

In[21]:= partialSum2[1, 100] // N
```

General: Exp[−31415.9] is too small to represent as a normalized machine number; precision may be lost.

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Out[21]= 100.

In[20]:= ListPlot[
          {Table[partialSum[n, 100], {n, 1, 200}], Table[partialSum2[n, 100], {n, 1, 200}]}]
```

Out[20]=