Last time: • If γ_1 & γ_2 homotopic $\Rightarrow$

$$\int_{\gamma_1} f = \int_{\gamma_2} f.$$

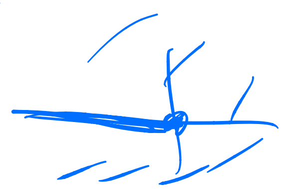
• Ω simply connected $\Rightarrow \oint_\gamma f = 0.$

• $\Rightarrow \mathbb{C} \setminus \{0\}$ not simp. conn. (since $\int_{C_1} \frac{1}{z}\,dz \neq 0$).

• If Ω simp conn, $1 \in \Omega$, $0 \notin \Omega$, $\log_\Omega(z) := \int_{1 \to z} \frac{1}{w}\,dw.$

(on $\mathbb{R} \cap B_{\frac{1}{2}}(1)$, $\log_\Omega = \log_{\mathbb{R}}$.

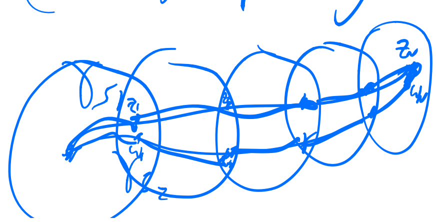
"$\exp(\log_\Omega(z)) = z$).



• "Principal branch" of log: $\Omega = \mathbb{C} \setminus (-\infty, 0]$,

if $z = re^{i\theta}$, $|\theta| < \pi$, $\log_\Omega(z) = \log_{\mathbb{R}} r + i\theta$,

$\log(zw) \neq \log z + \log w.$

Pf: 

$|s_1 - s_2| < \delta.$

Pf1: If F_j = prim on D_j,

$$\int_{z_j \to z_{j+1}} f = F_j(z_{j+1}) - F_j(z_j) \qquad F_j = F_{j+1} + C_j$$

on $D_j \cap D_{j+1}$

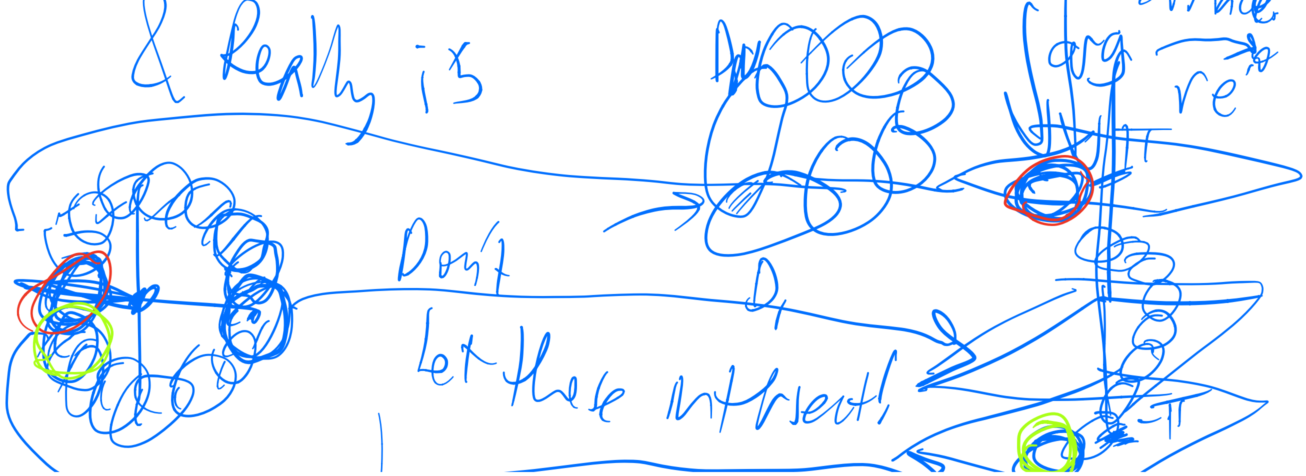
$$\underbrace{- \int_{w_j \to w_{j+1}} f}_{}$$

$= F_j(z_{j+1}) - F_j(w_{j+1})$. (Iterate until end pt meet).

Pf2: On overlaps $D_j \cap D_{j+1}$, two primitives, just adjust one of them so $C_j = 0$. So there is (by anal cont) a global primitive on $\bigcup D_j$.

log Riemann surface

& Really is



On each D_{3k_3} D_j, $\exists$ $\log_{D_j}$

Really can think about $\log(z)$:

$\mathbb{C}$

$\updownarrow$

gives 'all possible values if we look from "above",

$\mathbb{C}\setminus\{0\} \times \mathbb{R}$

& act of taking a "branch cut" specifies exactly which value log should take (no $\pm 2\pi i n$ ambiguity).

Point: Let γ be any (infinite) simple / injective curve: $0 \to \infty$.

$\gamma : [0,\infty) \to \mathbb{C}$

$(\forall R,\ \{t \mid |\gamma(t)| < R\}$ is bdd$\}$.

Ran J

log z s.t.

$\Omega = \mathbb{C} \setminus \gamma$

$\exp(\log) = z.$

Pf: Pick any $z_0 \in \Omega$

let $\log_\Omega = \int_{z_0 \to z} \frac{dw}{w}.$

γ is called a branch cut

Coupled exponentiation??

$\sqrt{2}^{\sqrt{\pi}}$?

$\frac{r_n}{s_n}$

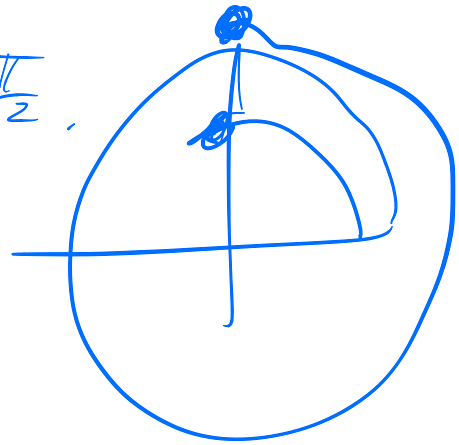
$\frac{p_n}{q_n}$

$\left(\frac{p_n}{q_n}\right)^{\frac{r_n}{s_n}} \to \sqrt{2}^{\pi}.$

integer root

$$i^i = \left(e^{\frac{\pi}{2}i}\right)^i = e^{-\frac{\pi}{2}}$$

$$= \left(e^{\frac{5\pi}{2}i}\right)^i = e^{-\frac{5\pi}{2}}$$



Over $\mathbb{C}$, $z^w := \exp\left(w \cdot \underline{\log z}\right)$

You give the branch.

If I shift $\log z$ by $2\pi i$

z^w shifts by $e^{w \cdot 2\pi i}$.

"Log of a function".

Simply conn

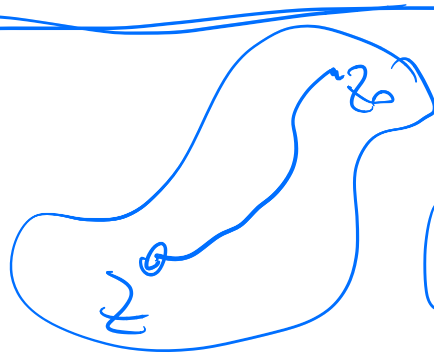
<u>Thm</u>: If $f : \Omega \to \mathbb{C}$

& $f \neq 0$. Then $\exists$ holo

$g(z)$ "$=\log f(z)$", s.t.

$e^{g(z)} = f(z)$ (Not unique). $g+2\pi i$

Pf:



Ω Fix $z_0 \in \Omega$

Let $g(z) := \displaystyle\int_{z_0 \to z} \underbrace{\overbrace{\boxed{\frac{f'(w)}{f(w)}}}^{\text{hol'c}}}_{\neq 0} \, dw$

Indep of path.

Nice hol'c function. (hol'c: $g(z+h) - g(z) = \int_{z \to z+h} \cdots$).

Look at: $\dfrac{d}{dz}\left(f(z)e^{-g(z)}\right) = f' \cdot e^{-g} + f \cdot e^{-g}\left(-\dfrac{f'}{f}\right)$

$= 0$.

$\left. f \cdot e^{-g} \right|_{z=z_0} = $ Const.

$\longrightarrow f(z_0) \cdot e^{-0} = f(z_0)$.

Need to shift g by a constant, $f(z_0) \neq 0$,

So $\exists$ choice of $w_0 = \log f(z_0)$, s.t. $e^{w_0} = f(z_0)$,

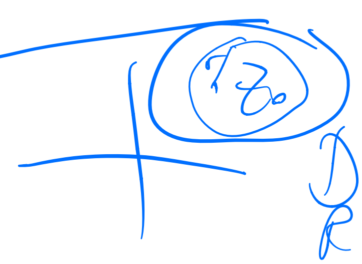
So g should really be: $\int_{z_0 \to z} \frac{f'}{f} + w_0 = g$,

$$f \cdot e^{-g}\Big|_{z_0} = f(z_0) \cdot e^{-(0 + w_0)} = 1,$$

§ Fourier series:

Thm: Let $f : D_R(z_0) \to \mathbb{C}$ hol.,

Then: $\sum_{n \geq 0} a_n (z - z_0)^n$



$\forall\, n \geq 0,\ \forall\, 0 < r < R,$

$$a_n = \frac{1}{2\pi r^n}\int_0^{2\pi} f(z_0 + re^{i\theta})\, e^{-in\theta}\, d\theta$$

Power series coefficients are <u>also</u> Fourier coefficients.

& if $n<0$, $\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})e^{-in\theta}\,d\theta = 0$.

Pf: $n \geq 0$.

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{\cancel{n!}}\,\frac{\cancel{n!}}{2\pi i}\int \frac{f(w)}{(w-z)^{n+1}}\,dw$$

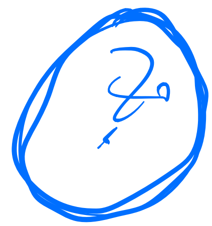
$w = z_0 + re^{i\theta} \to C_r(z_0)$, $0 < \theta < 2\pi$.

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{(re^{i\theta})^{n+1}}\, re^{i\theta} \cdot i\, d\theta,$$

$\underline{n<0:}$ $\quad \dfrac{1}{2\pi r^n}\displaystyle\int_0^{2\pi} f(z_0+re^{i\theta})\,e^{-in\theta}\,d\theta$

$$= \frac{1}{2\pi i}\int_{C_r(z_0)} \underbrace{f(w)\,(w-z)^{-n-1}}_{\text{hol'c}}\,dw = 0. \qquad -n-1 \geq 0:$$

<u>Cor (Mean Value Principle):</u>



$$\frac{1}{2\pi i}\int_{C_r(z_0)} f(w)\,dw = f(z_0).$$

$$\|$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta.$$

Cor: If $u: \mathbb{R}^2 \to \mathbb{R} \in C^2$, $\Delta u = 0$ (harmonic)

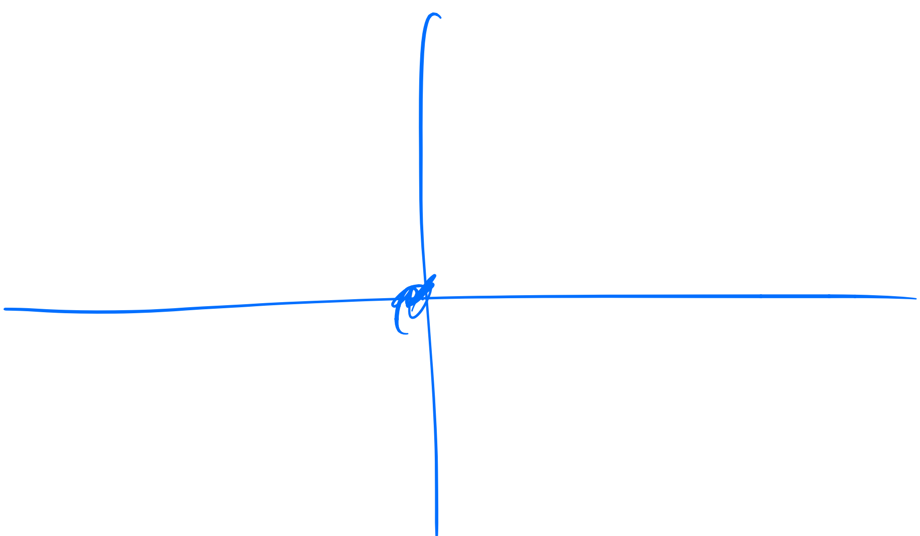
↑ Laplacian $\Delta = \partial_{xx} + \partial_{yy}$. Then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta$$

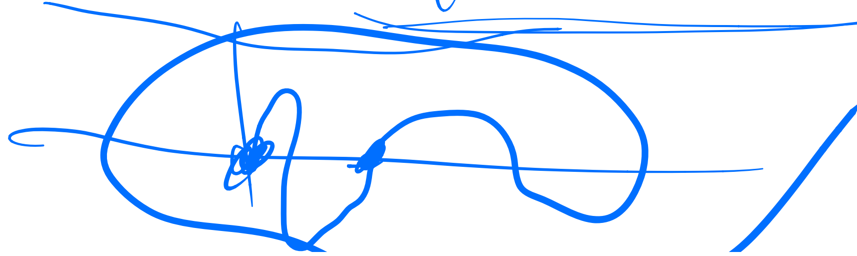
Pf: u harmonic $\Rightarrow \exists f$ holo, $\mathrm{Re}(f) = u$.

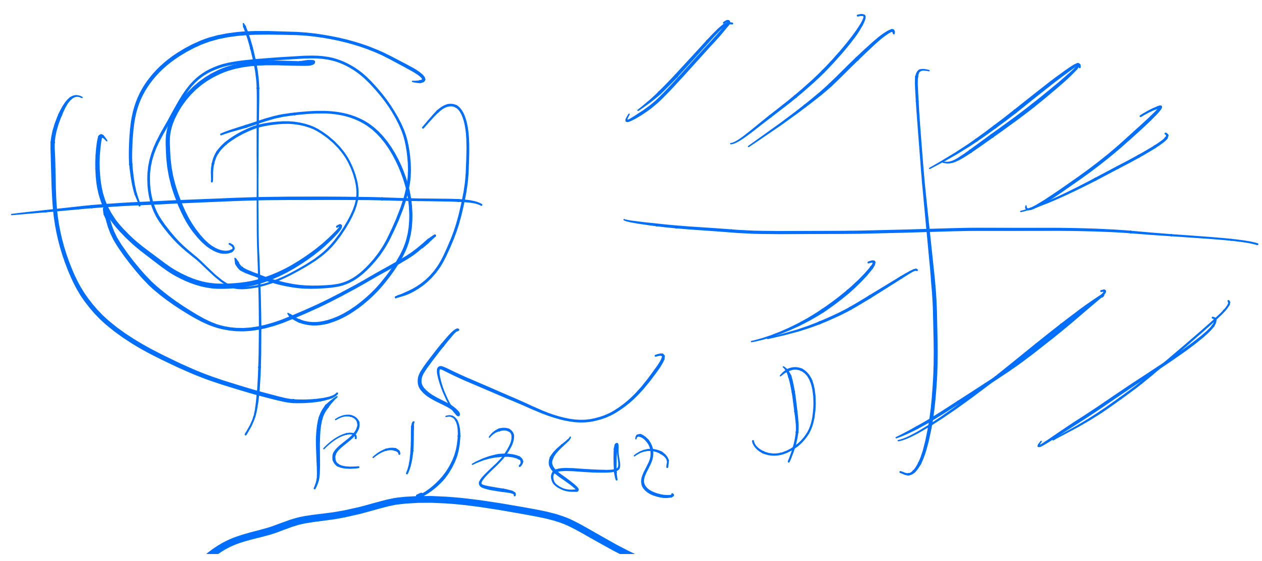
$$\boxed{\log z + \log(1-z)}$$

$$\log_{\Omega_1}(z) + \log_{\Omega_2}(1-z).$$



g : $e^{g(z)} = z(1-z) = f(z)$.

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