Last time: Arg Principle: $\int \frac{f'}{f} = \#\text{zeros} - \#\text{poles}$.

Rouché Thm: If $|f|>|g|$ on ∂D

$\Rightarrow$ # zeros of f in D

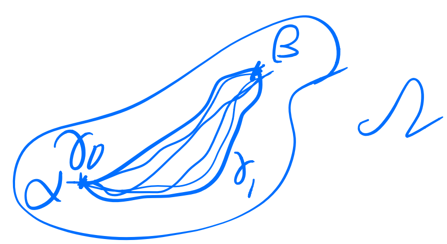
$=$ # zeros of $f+g$ in D.

Open Map Thm: $f(\text{open}) = \text{open}$.

Max Mod Principle: If $\exists z_0 \in \Omega$ s.t. $\forall z \in \Omega$,

$|f(z)| \le |f(z_0)| \Rightarrow f = \text{constant}$.

Def: $\gamma_0, \gamma_1 \subset \Omega$



$\gamma_0(0) = \gamma_1(0) = \alpha$, $\gamma_0(1) = \gamma_1(1) = \beta$ are homotopic

if: $\exists\ \gamma : \overset{s}{[0,1]} \times \overset{t}{[0,1]} \to \Omega$ jointly cont,

$\forall s \quad \gamma_s(0) = \alpha, \ \gamma_s(1) = \beta,$

& $\gamma|_{s=0} = \gamma_0$, $\gamma|_{s=1} = \gamma_1$

<u>Thm</u>: If f hol on Ω & $\gamma_0, \gamma_1 \subset \Omega$ are homotopic, then $\int_{\gamma_0} f = \int_{\gamma_1} f$.

<u>Pf</u>: Let $\gamma : \underbrace{[0,1]\times[0,1]}_{\text{compact}} \to \Omega$ be a homotopy.

<u>Claim</u>: image $\gamma([0,1]\times[0,1]) =$ cpt. $= K$.

So $\exists \varepsilon > 0$ s.t.

$$d(\underset{\text{closed}}{\Omega^c}, \underset{\text{cpt}}{K}) > 3\varepsilon > 0 \quad \text{i.e.}$$

$\Omega^c \cap K = \emptyset$,

$\forall w \in K$, $D_\varepsilon(w) \subset \Omega$

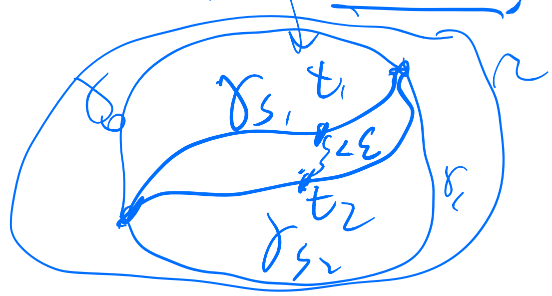
<u>Fix this $\varepsilon > 0$</u>

γ is jointly cont (compact $\rightarrow$)

$\Rightarrow$ uniformly cont.

Given $\varepsilon > 0$ $\exists \delta > 0$: $\forall\, |s_1 - s_2| < \delta$, $\forall\, |t_1 - t_2| < \delta$,

$$\left| \gamma_{s_1}(t_1) - \gamma_{s_2}(t_2) \right| < \varepsilon.$$



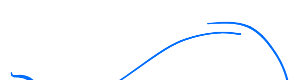
in particular, $\forall\, |s_1 - s_2| < \delta$, $\forall\, t \in [0,1]$,

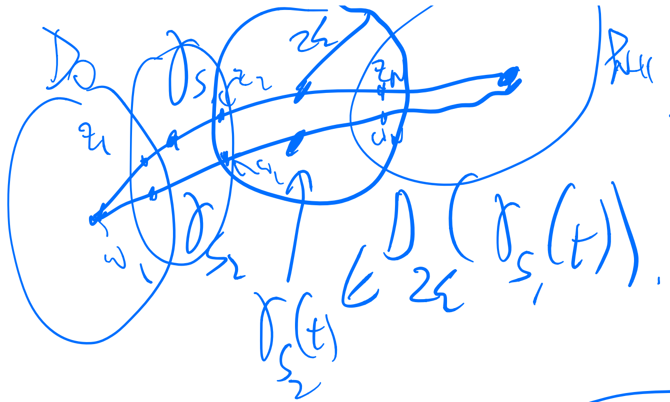
$$\left| \gamma_{s_1}(t) - \gamma_{s_2}(t) \right| < \varepsilon. \quad (\text{letting } t_1 = t_2 = t).$$

Claim: $\displaystyle\int_{\gamma_{s_1}} f = \int_{\gamma_{s_2}} f$ (for $|s_1 - s_2| < \delta$).

(so in $\frac{1}{\delta}$ steps, walk from $\displaystyle\int_{\gamma_0} = \int_{\gamma_\delta} = \cdots = \int_{\gamma_1}$).

For each $t \in [0,1]$, consider $D_{2\varepsilon}(\gamma_{s_1}(t)) \subset \Omega$





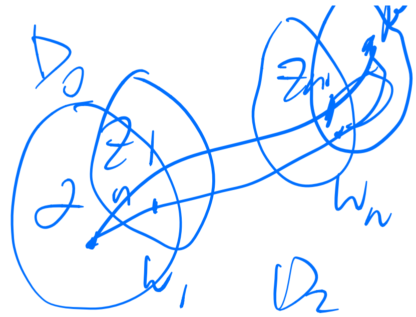
So as t ranges, these disks cover $\gamma(\underbrace{\{s_1, s_2\} \times [0,1]}_{\text{cpt}})$ cpt, so only need finitely many disks, $D_0, \dots, D_{N+1}$.

On D_0, f is hol., so has a primitive F_0. On D_1, f is hol. & has a primitive F_1. On $D_0 \cap D_1$, $F_0' = F_1' \Rightarrow F_0 = F_1 + C$.

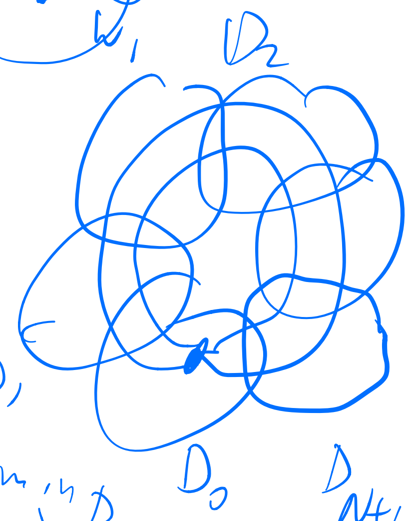
Replace F_1 by $F_1 - C$ (still a primitive on D_1) & now F_1 is an analytic cont of F_0 to $D_0 \cup D_1$. Continue extending analytically to last disk $\Rightarrow$ have primitive

$\square$

$$\int_{\gamma_{s_1}} f = \boxed{\int_{\alpha\to z_1}} + \int_{z_1\to z_2} + \dots + \int_{z_N\to\beta},$$



$$\int_{\gamma_{s_2}} f = \int_{\alpha\to w_1} + \int_{w_1\to w_2} + \dots + \int_{w_N\to\beta}.$$



F_0 primitive on D_0, prim. in D_1

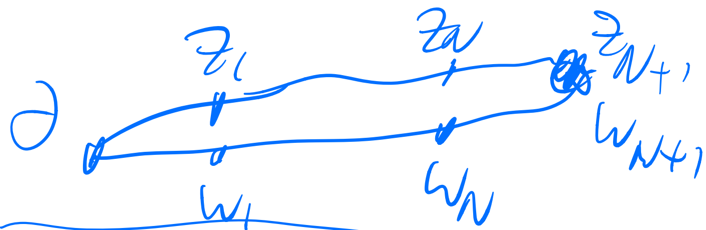
$$\int_{\alpha\to z_1} f = F_0(z_1) - F_0(\alpha), + \int_{z_1\to z_2} f = F_1(z_2) - F_1(z_1)$$

$$-\int_{\alpha\to w_1} f = \left(F_0(w_1) - F_0(\alpha)\right) - \left(\int_{w_1\to w_2} f = F_1(w_2) - F_1(w_1)\right)$$

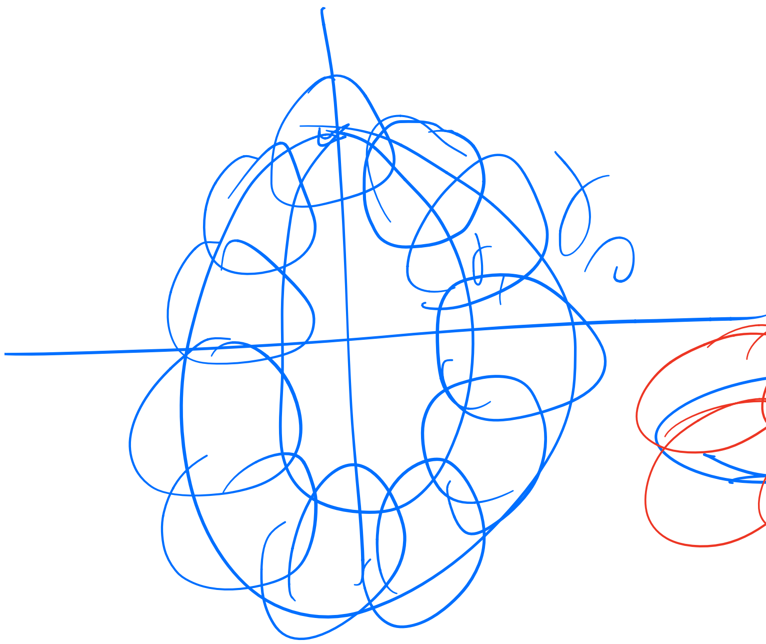
$$F_0(z_1) - \cancel{F_0(\alpha)} + F_1(z_2) - F_1(z_1) = C \qquad = F_1(z_2) - F_1(w_2)$$

$$-F_0(w_1) + \cancel{F_0(\alpha)} - F_1(w_2) + F_1(w_1). = -C$$

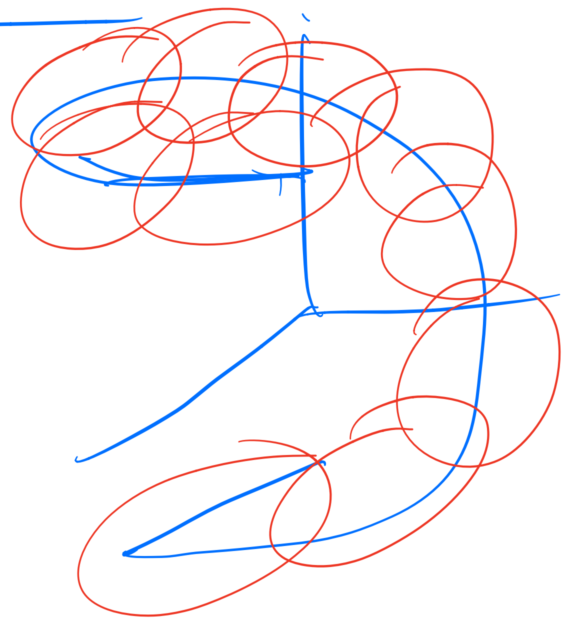
Continuing $\sum_{\alpha\to z_1\to z_2\to\cdots z_N} - \sum_{\alpha\to w_1\to\cdots\to w_N} = F_N(z_{N+1}) - F_N(w_{N+1})$



$= 0.$



$\mathbb{C}\setminus\{0\} = \Omega$

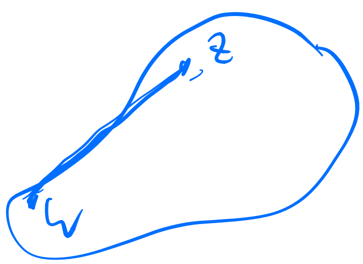


Def: Ω is simply connected if

$\forall\ \gamma_0, \gamma_1 \subset \Omega$ (with same end pts), $\alpha \to \beta$, they are homotopic.

<u>Ex:</u> Any Ω convex is simp. conn.

($\forall\ z, w \in \Omega$, $\forall t \in [0,1]$) $\underline{z \cdot t + w \cdot (1-t) \in \Omega}$.



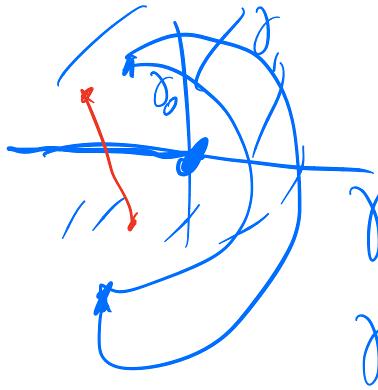
Given $\gamma_0, \gamma_1 : \alpha \to \beta$,

$\gamma : [0,1] \times [0,1] : (s,t) \mapsto$

$$(1-s)\gamma_0(t) + s \cdot \gamma_1(t) \in \Omega.$$

is jointly cont, is a homotopy.

<u>Ex:</u> "split plane" $\mathbb{C} \setminus \{(-\infty, 0]\}$.

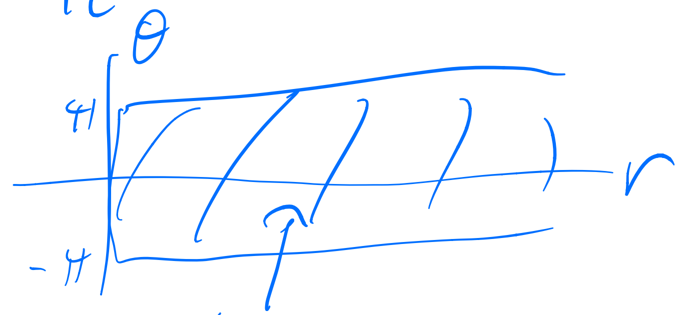


Simply conn. (not convex!).

$\gamma_0(t) = r_0(t) e^{i\theta_0(t)}$ $\quad |\theta| < \pi$.

$\gamma_1(t) = r_1(t) e^{i\theta_1(t)}$.

$\underbrace{(0,\infty)}_{r} \times \underbrace{(-\pi,\pi)}_{\theta}$



convex.

Let

$$\gamma_s(t) = \underbrace{\left[(1-s)r_0(t) + s\,r_1(t)\right]}_{\text{or } r_0^{(1-s)}(t)\cdot r_1^{s}(t)} e^{i\left[(1-s)\theta_0(t) + s\,\theta_1(t)\right]}.$$

Thm: If f hol. on Ω simp conn.

$\Rightarrow$ f has primitive. $\Rightarrow \oint_\gamma f = 0$.

pf: let $F(z) := \int_{z_0 \to z} f$ indep of path!

$F'(z) = f(z)$

(zoom in near z, $F(z+h)-F(z) = \int_{z\to z+h} f(w)\,dw$).

Thm: $\mathbb{C}\setminus\{0\}$ is not simply connected.

pf: If it was, $\oint_{\text{unit circle}} \frac{dw}{w} = 0$. ~~It~~

$= 2\pi i$.

(Proving topological statement using analysis!).

Remark: Can be simply conn w/o being connected.

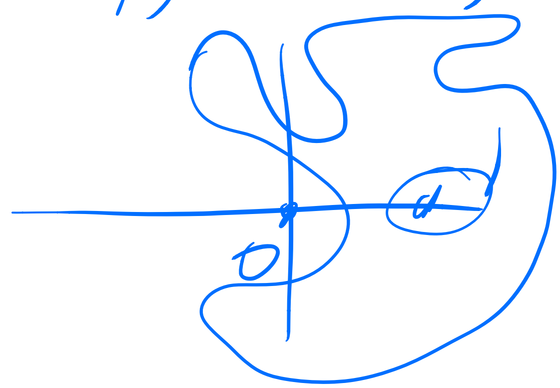
Assume $\gamma_0, \gamma_1 : \alpha \to \beta$. Question of homotopy is in connected component.

(vacuous otherwise, since no curves exist between disconnected components).

§ Complex log: want $\log_{\mathbb{C}}(z_1 z_2) = \log_{\mathbb{C}}(z_1) + \log_{\mathbb{C}} z_2$

& $\log_{\mathbb{C}}(re^{i\theta}) \overset{?}{=} \log_{\mathbb{R}} r + i\theta$, $\theta \in (-\pi, \pi)$. Can't have both!

Thm: Let Ω be simply connected, $1 \in \Omega$, $0 \notin \Omega$,



Then $\exists\, F(z)$ "$= \log_{\Omega} z$" hol'c on Ω s.t.

(1) $F'(z) = \frac{1}{z}$, (2) $e^{F(z)} \overset{\checkmark}{=} z$

(3) $F(r)\Big|_{\underbrace{D_\varepsilon(1) \cap \mathbb{R}}_{\Omega \supset}} = \log_{\mathbb{R}} r$ ✓, $\exists \varepsilon > 0$

Pf: let $F(z) := \int_{1 \to z} \frac{dw}{w}$. well-def indep of path

(1) ✓

for (2) ✓, look at: $\frac{d}{dz}\left(z \cdot e^{-F(z)}\right) = 1 \cdot e^{-F(z)} + z \cdot e^{-F(z)}\left(-\cancel{F'(z)}\right) = 0$.

$\Rightarrow \left. z e^{-F(z)} \right|_{z=1} = \text{const.} = 1 \cdot e^{-F(1)} = 1$

with $F(1) = 0$.

(3): near 1, $r \in \mathbb{R}$, take γ = horiz.

$F(z) = \int_1^r \frac{dx}{x} = \log_{\mathbb{R}} r$.

Def: When $\Omega = \mathbb{C} \setminus \{(-\infty, 0]\}$,

— simp conn, $\not\ni 0$

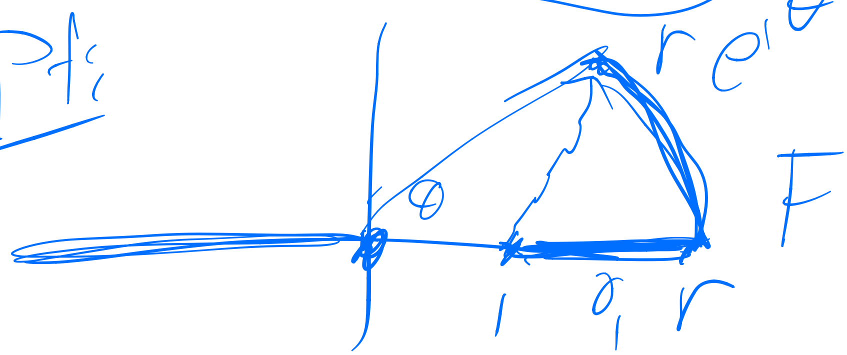
$\log_\Omega$ = "principal branch of log"

$\log_\Omega = \log$.

Cor: For $z = re^{i\theta} \in \mathbb{C} \setminus (-\infty, 0]$, $\theta \in (-\pi, \pi)$,

$$\log z = \boxed{\log r + i\theta}.$$

Pf:



$$F(z) = \int_1^{re^{i\theta}} \frac{dw}{w}.$$

$$\int_{\gamma} = \int_{\gamma_1} + \int_{\gamma_2} = \qquad \int_{\gamma_1} = \int_1^r \frac{dx}{x} = \boxed{\log_R r}$$

$\gamma_1 : z = x,\ 1 < x < r$

$$\int_{\gamma_2} = \int_0^{\theta} \frac{1}{\cancel{re^{i\varphi}}} \cancel{re^{i\varphi}}\, i\, d\varphi = \boxed{i\theta.}$$

$\gamma_2 \leftarrow z = re^{i\varphi},\ 0 < \varphi < \theta$
