Last time: $f(z_0)=0 \Rightarrow f(z) = (z-z_0)^{\overset{\text{order}}{N}}\, \underset{\neq 0}{g(z)}$ $\forall z \in U \ni z_0$.

- if f has a pole at z_0, $\forall z \in U \setminus \{z_0\}$, $f(z) = \underbrace{\frac{a_{-N}}{(z-z_0)^N} + \dots + \frac{\boxed{a_{-1}}}{(z-z_0)}}_{\text{principal part}} + \underset{\text{hol'c}}{G(z)}$, $a_{-N} \neq 0$, residue a_{-1}

- **Thm:** $\operatorname{Res}_{z_0} f = \frac{1}{(N-1)!}\left(\frac{d}{dz}\right)^{N-1} \Big[(z-z_0)^{\overset{\text{order}}{N}} f\Big]\Big|_{z\to z_0}$

$$\to \begin{cases} \infty & N \text{ too small} \\ 0 & N \text{ too big} \\ \text{else} & N \text{ just right} = \text{order} \end{cases}$$

Residue Thm: Let $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ hol'c & have pole at z_0. Let $\Omega \supset \overline{D} \ni z_0$.

Then
$$\boxed{\frac{1}{2\pi i}\int_{C=\partial D} f(z)\,dz = \operatorname{Res}_{z_0} f}$$

Pf: $\int_\gamma f(z)\,dz = 0$.

Send $\varepsilon \to 0$

$$\boxed{C} = \int_{\partial B_\varepsilon(z_0)} f(z)\,dz$$

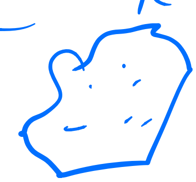
$$\forall \varepsilon,\quad \int_{\partial B_\varepsilon} f(z) = \cancel{\int_{\partial B_\varepsilon} \frac{a_{-N}}{(z-z_0)^N}} + \dots + \boxed{\int_{\partial B_\varepsilon} \frac{a_{-1}}{(z-z_0)}} + \int_{\partial B_\varepsilon} G(z)$$

has primitive $\dfrac{(z-z_0)^{-N+1}}{(1-N)}$

hol in B_ε, has primitive

$$= a_{-1} \cdot 2\pi i$$

Cor: If $R \subset \Omega$, f has poles $z_0, z_1, \dots$ in R, & ∂R = toy contour. R



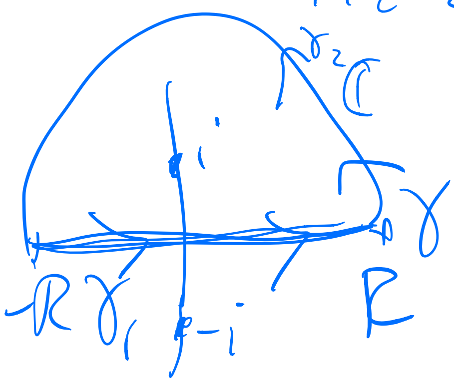
$$\Rightarrow \frac{1}{2\pi i}\int_{\partial R} f(z)\,dz = \sum_{j=0}^{K} \mathrm{Res}_{z_j} f.$$

Ex: $I = \int_{\mathbb{R}} \frac{dx}{1+x^2} = \arctan x \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$

$\boxed{2=\pi=i=1}$

$f(z) = \frac{1}{1+z^2}$ ← poles at $z = \pm i$

$f(z) = \frac{1}{(z+i)(z-i)}.$



By Residue Thm

$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \mathrm{Res}_i f = \frac{1}{0!}\left(\frac{d}{dz}\right)^0 \left\{(z-i)f\right\}\Big|_{z\to i} = \frac{1}{2i}$

$I = \int_{\gamma_1} f(z)\,dz$

$\left|\int_{\gamma_2} f(z)\,dz\right| = \left|\int_0^\pi \frac{1}{1+R^2e^{2i\theta}}\, iRe^{i\theta}\,d\theta\right|$

$\gamma_2 \leftarrow z = Re^{i\theta}\quad 0<\theta<\pi$

$z' = Re^{i\theta}(i)$

$\leq R\int_0^\pi \frac{1}{R^2-1}\,d\theta \to 0.$

$\frac{1}{2\pi i} I = \frac{1}{2i}.$ ✓.

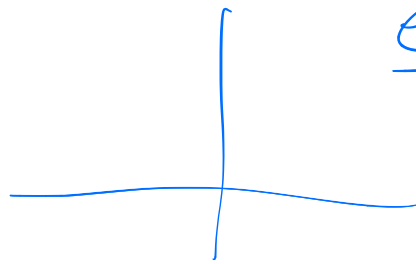
Ex: Laplace transform of $\frac{1}{1+e^x}$.

i.e. $\boxed{0<a<1}$ $\displaystyle\int_{\mathbb{R}} \frac{e^{ax}}{1+e^x}\,dx = ?$ $a\in\mathbb{R}$.

$\frac{1}{1+e^x}$.

At $+\infty$, $\frac{e^{ax}}{e^x} = e^{(a-1)x} \to 0$ $a-1<0$.

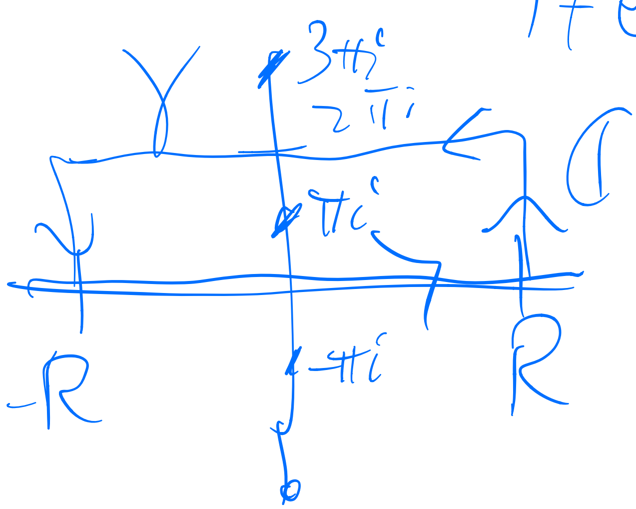
$\frac{e^{ax}}{1+e^x}$



At $-\infty$? $e^{ax}\to 0$ as $x\to-\infty$. $a>0$.

$$f(z) = \frac{e^{az}}{1+e^z}.$$

has poles $e^z=-1$, $z=(2n+1)\pi i$



Notice: $f(z+2\pi i) = \dfrac{e^{az}}{1+e^{z}} \cdot e^{2\pi a i}$

quasi invariance under shift by $2\pi i$ suggests γ.

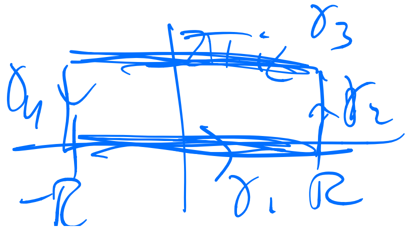
$$\frac{1}{2\pi i}\int_{\gamma} f(z)\,dz = \operatorname{Res}_{\pi i} f = -e^{a\pi i}$$

Order of pole? It's $N=1$.

$$\lim_{z\to\pi i}{}^{?}\ \frac{1}{0!}\left(\frac{d}{dz}\right)^{0}\frac{(z-\pi i)\,e^{az}}{1+e^{z}} \longrightarrow \frac{e^{a\pi i}}{-1}.$$

$e^{az} \to$ limit.

Observe: $\displaystyle\lim_{z\to\pi i}\frac{e^{z}-e^{\pi i}}{z-\pi i} = \left(\frac{d}{dz}e^{z}\right)\Big|_{\pi i} = e^{\pi i} = -1.$



$$\int_{\gamma_1} \to I \quad (\text{as } R\to\infty).$$

$$\int_{\gamma_3} = -\int_{\gamma_3^-} \cancel{\phi} = -\int_{-R}^{R} \frac{e^{a(2\pi i + x)}}{1+e^{x+2\pi i}} \cdot 1 \cdot dx$$

$\gamma_3^- \leftarrow z = 2\pi i + x,\ -R<x<R$

$$\Rightarrow -\underbrace{e^{a2\pi i}} \cdot I.$$

$$\left|\int_{\gamma_2}\right| = \left|\int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{R+iy}}\, dy\right| \le e^{aR}\int_0^{2\pi} \frac{1}{e^R-1}\, dy. \xrightarrow{a<1} 0.$$

$\gamma_2 \leftarrow z = R+iy,\ 0<y<2\pi$

Evaluate $\int_{\gamma_4}$,

$$\left|\int_{\gamma_4}\right| = \left|-\int_{\gamma_4^-}\right| = \left|-\int_0^{2\pi} \frac{e^{a(-R+iy)}}{1+e^{\cancel{-R+iy}}(R+iy)} \cdot i \cdot dy\right|$$

$\gamma_4^- \leftarrow z = -R+iy,\ 0<y<2\pi$

$$\le e^{-aR}\int_0^{2\pi} \frac{1}{\cancel{e^{-R}-1}\ 1-e^{-R}}\, dy. \xrightarrow{a>0} 0.$$

$$\frac{1}{2\pi i}\int_\gamma \Rightarrow \frac{1}{2\pi i}I + \frac{1}{2\pi i}I\left(-e^{a2\pi i}\right) = -e^{a\pi i}$$

$$I = \frac{+e^{a\pi i}\cdot 2\pi i}{-1+e^{a2\pi i}} \qquad \int_{-\infty}^{\infty}\frac{e^{ax}\,dx}{1+e^{x}}$$

$$= \frac{2\pi i}{e^{a\pi i}-e^{-a\pi i}} = \boxed{\frac{\pi}{\sin a\pi}}.$$

Ex: Fourier transform of $\frac{1}{\cosh x}$.

$$\cosh x = \frac{e^{x}+e^{-x}}{2} = \cos ix$$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

Want: $\xi\in\mathbb{R}$

$$\int_{-\infty}^{\infty}\frac{e^{-2\pi i x\xi}}{\cosh(x\pi)}\,dx.$$

$$f(z) = \frac{2 \cdot e^{-2\pi i \xi z}}{e^{(z+1)} + e^{-(z+1)}}$$

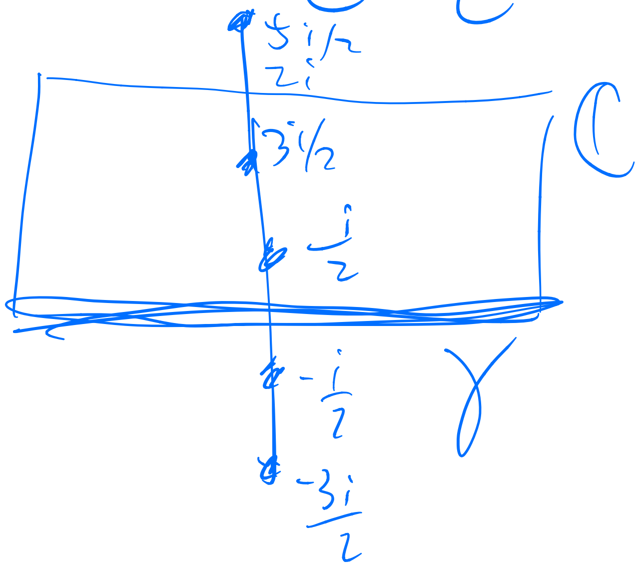
poles when

$$e^{\pi z} = -e^{-z\pi}.$$

$$e^{2z\pi} = -1.$$

$$\cancel{2}\pi z = \frac{(2n+1)}{2}\pi i$$

$$\pm \frac{i}{2}, \pm \frac{3}{2} i, \ldots$$



Let's shift by $2i$

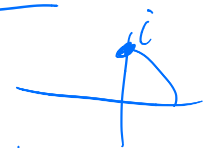
$$f(z+2i) = \boxed{\frac{2 \cdot e^{-2\pi i \xi z}}{e^{\pi z} + e^{-\pi z}}} \cdot \underbrace{e^{-2\pi i \xi 2i}}_{e^{4\pi \xi}}$$

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \mathrm{Res}_{\frac{i}{2}} f + \mathrm{Res}_{\frac{3i}{2}} f.$$

$$\left(z - \frac{i}{2}\right) \frac{2e^{-2\pi i \xi z}}{e^{\pi z} + e^{-\pi z}} \longrightarrow ? \text{ as } z \to \frac{i}{2}.$$

//

$$\text{''} \quad \boxed{\frac{\left(z-\frac{i}{2}\right)\cdot 2\cdot e^{-2\pi i \xi z}\cdot e^{\pi z}}{e^{2\pi z}-e^{2\pi(i/2)}}}$$



$$\frac{d}{dz}\left(e^{2\pi z}\right)\Big|_{\frac{i}{2}} = \lim_{z\to\frac{i}{2}} \frac{e^{2\pi z}-e^{2\pi(i/2)}}{z-i/2}$$

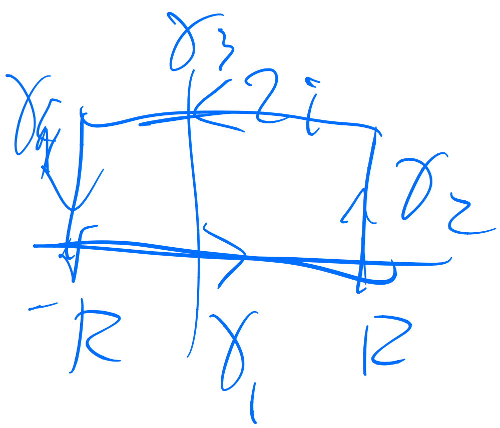
$$\text{''}$$

$$e^{2\pi\left(\frac{i}{2}\right)}\cdot 2\pi = \boxed{-2\pi.}$$

$$\boxed{\operatorname{Res}_{i/2} f = -\frac{1}{\pi}\cdot e^{-2\pi i \xi\frac{i}{2}}\, e^{\pi\frac{i}{2}}}$$

Exercise: $\boxed{\operatorname{Res}_{3i/2} f = -\frac{e^{3\pi\xi}}{\pi i}}$ $= \frac{e^{\pi\xi}}{\pi i}$

$$\frac{1}{2\pi i}\int_\gamma f = \boxed{\frac{e^{\pi\xi}}{\pi i} - \frac{e^{3\pi\xi}}{\pi i}}$$



$$\int_{\gamma_1} \to I.$$

$$\int_{\gamma_3} = -\int_{\gamma_3^-} e^{4\pi\xi} f(x)\,dx \to -e^{4\pi\xi}\cdot I.$$

$\gamma_3^- : z = 2i + x, \ -R < x < R$

$$\left|\int_{\gamma_2}\right| = \left|\int_0^2 \frac{e^{-2\pi i\xi(R+iy)}}{e^{R+iy}+e^{-R-iy}}\cdot 2\cdot i\,dy\right|$$

$\gamma_2 : z = R + iy, \ 0 < y < 2$

$$\le \int_0^2 \frac{e^{2\pi\xi y}}{e^R - 1}\,dy \le \frac{e^{4\pi\xi}}{e^R}\cdot K \to 0.$$

& $\int_{\gamma_4} \to 0$ (check).

$$\frac{1}{2\pi i}\left(\int_{\gamma_1} + \int_{\gamma_3}\right) \to \frac{1}{2\pi i}\left(I\left(1 - e^{4\pi\xi}\right)\right)$$

//

$$\frac{1}{\pi i}\left(e^{\pi\xi} - e^{3\pi\xi}\right),$$

$$I = \frac{2\cancel{e^{2\pi\xi}}\cancel{\left(e^{-\pi\xi} - e^{\pi\xi}\right)}}{\cancel{e^{2\pi\xi}}\underbrace{\left(e^{-2\pi\xi} - e^{2\pi\xi}\right)}_{\cancel{\left(e^{-\pi\xi}-e^{\pi\xi}\right)}\left(e^{-\pi\xi}+e^{\pi\xi}\right)}}.$$

$$\int_{\mathbb{R}} \frac{e^{-2\pi i \xi x}}{\cosh \pi x}\,dx = \boxed{\frac{1}{\cosh \pi\xi}}$$

own Fourier transform!

(like $e^{-\pi x^2}$).