Last time: If f hol'c $\Omega \to \mathbb{C}$
$D \subset \Omega$,

$\forall z \in D$ $\forall n \geq 0$, $f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}} dw$.
$C = \partial D$

Cor (Cauchy Ineq): If $\overline{D_R(z)} \subset \Omega$, $|f^{(n)}(z)| \leq \frac{n!}{R^n} \sup_C |f|$.

Cor (Liouville): f entire & bdd $\Rightarrow f \equiv c$.

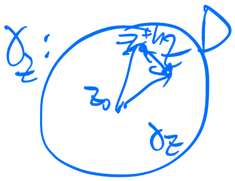
Cor FTA: $f \in \mathbb{C}[x] \Rightarrow f = a(z-a_1)\cdots(z-a_n)$.

All followed from baby Cauchy aka Goursat:
If f hol'c on $\Delta \subset \Omega \Rightarrow \int_\Delta f = 0$.

Goursat Converse (Morera's Thm): If
$f : \Omega \xrightarrow{\text{open, conn}} \mathbb{C}$ cont. & $\forall \Delta \subset \Omega$,
$\int_\Delta f = 0 \Rightarrow f$ hol'c.

Pf: Let $D \subset \Omega$ be a disk. Claim: f has a primitive in D. If so, $f \nearrow F \searrow F'=f \searrow F''=f'$.

Let γ_z:



Let $F(z) = \int_{\gamma_z} f(w)\,dw$. <u>Claim:</u> $F'=f$

Look at $F(z+h)-F(z) = \int_{z \to z+h} f(w)\,dw$ f cont $\Rightarrow$ $f(w) = f(z)+o_{w\to z}(1)$.

As $h\to 0$, $\Rightarrow w \to z$.

$$= \underbrace{\int_{z\to z+h} f(z)\,dw}_{f(z)\cdot h} + \underbrace{\int_{z\to z+h} o_{h\to 0}(1)\,dw}_{o_{h\to 0}(|h|)}$$

$\boxed{\text{Exercise: } A\to B \Rightarrow C\to D. \quad \Rightarrow o_{C\to D}(X) = o_{A\to B}(X).}$

$$\frac{1}{h}\left(F(z+h)-F(z)\right) = f(z) + o_{h\to 0}\left(\frac{|h|}{|h|}\right) \to 0 \text{ as } h\to 0.$$

$\boxed{X = O(Y) \quad | \quad \cancel{O(X)=X}}$

Immediate Application:

Over $\mathbb{R}$, Weierstrass Approx <u>Thm:</u> Every $f \in C([0,1])$ compact. B

<u>Uniformly</u> approx by <u>Polynomials</u>

So over $\mathbb{R}$,  is uniformly approximable by polynomials.

Thm: Let $f_n : \overset{\text{open, conn}}{\Omega} \to \mathbb{C}$ be holo

& let $f_n \to f$ uniformly on compacta,

i.e. $\forall K \overset{\text{cpt}}{\subset} \Omega \ \forall \varepsilon > 0 \ \exists N \ \forall n > N,$

$\forall z \in K, \ |f_n(z) - f(z)| < \varepsilon.$ 

Then f is holo.

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pf: Let $D \subset \Omega$ be a disk & let

$K \to \Delta \subset D$ Then $0 = \int_\Delta f_n$ (Goursat).

$\Rightarrow \int_\Delta f = 0$ (& f = cont (uniformly cont $\Rightarrow$ cont)) $\xrightarrow{\text{Morera}}$ holo.

i.e. over $\mathbb{C}$, continuity is not sufficient to be approx by polynomials (& analytic functions)

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§ Singularities. $z_0 \in \Omega$.

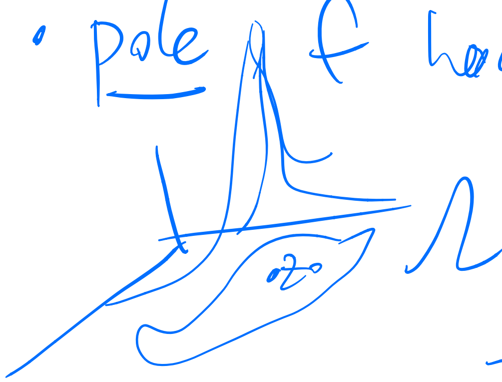
Let $f \stackrel{\text{holo}}{:} \Omega \setminus \{z_0\} \to \mathbb{C}$.

Then z_0 is an isolated singularity.

- removable. $\dfrac{z(\cancel{1+z})}{\cancel{1+z}}$ has remov sing at $z=-1$.

  i.e. $\exists\ g \stackrel{\text{holo}}{:} \Omega \to \mathbb{C}$ & $g|_{\Omega\setminus\{z_0\}} \equiv f$.

- pole. f has a pole at z_0 if

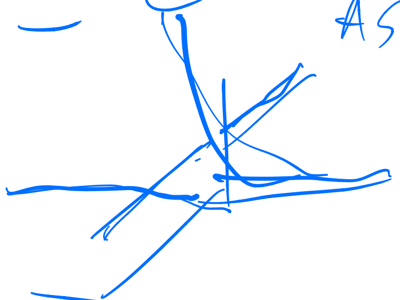


  i.e. if $1/f$ has a removable sing at z_0, $1/f(z_0) = 0$.

- essential: OTHER.

Ex $f(z) = \frac{1}{z}$.

Ex: $e^{1/z}$



As $z \to 0^+$, $f \to \infty$. As $z \to 0^-$, $f \to 0$.

As $z = it$, $|f| = 1$.

$1/f = \begin{cases} z & z \neq 0, \\ \text{undef at } z = 0 \end{cases}$

$\frac{1}{f}$ has no removable sing, not $\to 0$.

$$f(w) = f(z) + \underline{o_{w \to z}(1)}.$$

$\Updownarrow$

$$f(w) = f(z) + E,$$

error

$E = f(w) - f(z)$, $< \varepsilon$ when $|h| < \delta$.

$$|\underline{f(w) - f(z)}| \to 0 \text{ as } \underline{w \to z}.$$

$$f(w) = f(z) + \underline{o_{h \to 0}(1)} \Rightarrow |\underline{f(w) - f(z)}| \to 0 \text{ as } h \to 0.$$

<u>Zeros:</u> <u>Thm:</u> If $f: \Omega \to \mathbb{C}$ hol'c has a zero at z_0, then ($f \not\equiv 0$)

$\exists\, U \ni z_0$, $\exists\, g: U \to \mathbb{C}$ hol'c, $g \neq 0$,

$\exists!\, N \in \mathbb{N}$ s.t. $\forall z \in U$, $f(z) = (z-z_0)^N \cdot g(z)$.

<u>Pf:</u> Since $f \not\equiv 0$, z_0 is isolated, i.e.

$\exists\, U \ni z_0$ ($D_R(z_0)$) s.t. $f(z) = 0$ in $U \Rightarrow z = z_0$.

i.e. $f \neq 0$ in U, except at z_0. $\forall z \in U$,

$f(z) = \sum_{n \geq N} a_n \underline{(z-z_0)^n}$, let N = least

n s.t. $a_n \neq 0$. (If all $a_n = 0 \Rightarrow f \equiv 0$).

So $f(z) = (z-z_0)^N \underbrace{\left(a_N + a_{N+1}(z-z_0) + a_{N+2}(z-z_0)^2 + \dots\right)}$

g is cont, $g \to a_N \neq 0$ as $z \to z_0$ $\qquad$ $g \Rightarrow g$ holo.

So $\forall z \in U$, $|g| \geq |a_N|/2 > 0$, $\Rightarrow g \neq 0$ on U.

Unique? Say $N_1 > N_2$ &

$f = (z-z_0)^{N_1} g_1 = (z-z_0)^{N_2} g_2$.

& $g_1, g_2 \neq 0$, $\Rightarrow \underbrace{(z-z_0)^{N_1-N_2}}_{\to 0 \text{ as } z \to z_0} \cdot g_1 = g_2$. $\times$

& g unique.

Def. If f ($\not\equiv 0$, holo) vanishes at z_0,

& $f = (z-z_0)^N \cdot \overset{\neq 0}{g}$ near z_0, then $N =$ "order of zero".

If $N=1$, z_0 is a "simple zero" of f.

Struct Thm of Poles: If $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ hol; & has pole at z_0. ($\frac{1}{f} = 0$ at z_0)

Then $\exists U \ni z_0$, $\exists g$ on U, $g \neq 0$, $\exists! N$ s.t. $\forall z \in U$,

$$f(z) = (z-z_0)^{-N} \cdot g(z).$$

Pf: $\frac{1}{f}$ has a zero $\Rightarrow$ $\frac{1}{f} = (z-z_0)^N \cdot \underset{\neq 0}{g}$ in U.

$\Rightarrow f = (z-z_0)^{-N} \underset{\neq 0}{\left(\frac{1}{g}\right)}$.

Def: N is order of pole at z_0

$N=1 \Rightarrow$ "simple pole".

Thm ("Principal part expansion"):

If $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ has a pole at z_0 of order N, then near z_0,

$$f(z) = \underbrace{\frac{a_{-N}}{(z-z_0)^N} + \frac{a_{-N+1}}{(z-z_0)^{N-1}} + \dots + \frac{a_{-1}}{(z-z_0)}}_{\text{"principal part"}} + \underbrace{G(z)}_{\text{hol } \text{on } U}.$$

Pf: $f = \frac{1}{(z-z_0)^N}\left[A_0 + A_1(z-z_0) + A_2(z-z_0)^2 + \dots\right]$, ...

Def: a_{-1} = "residue" of f at z_0.

Rmk: z^n $\forall n \in \mathbb{Z} \setminus \{-1\}$, has nice primitive: $\frac{z^{n+1}}{n+1}$.

How to "pick off" a_{-1} from f?

<u>Thm</u>: $\boxed{\operatorname{Res}_{z_0} f = \frac{1}{(N-1)!}\left(\frac{d}{dz}\right)^{N-1}\left[(z-z_0)^N f\right]\Big|_{z=z_0}}$.

<u>pf</u>: $f = \frac{a_{-N}}{(z-z_0)^N} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$

$$\left(\frac{d}{dz}\right)^{N-1}(z-z_0)^N f = \underbrace{\cancel{a_{-N}} + \cdots + a_{-2}\cancel{(z-z_0)^{N-2}}} + a_{-1}\cancel{(z-z_0)^{N-1}} + a_0(z-z_0)^N + \cdots$$

$(N-1)!$

$* (z-z_0)$

<u>Def</u>: $f : \Omega \setminus \{z_0, z_1, z_2, \ldots\} \to \mathbb{C}$.

has poles at $z_0, z_1, \ldots$ then $f =$ "meromorphic".

Exercise:

$$\int e^{1/z}\,dz = ?$$