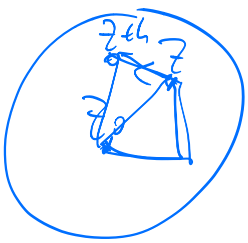
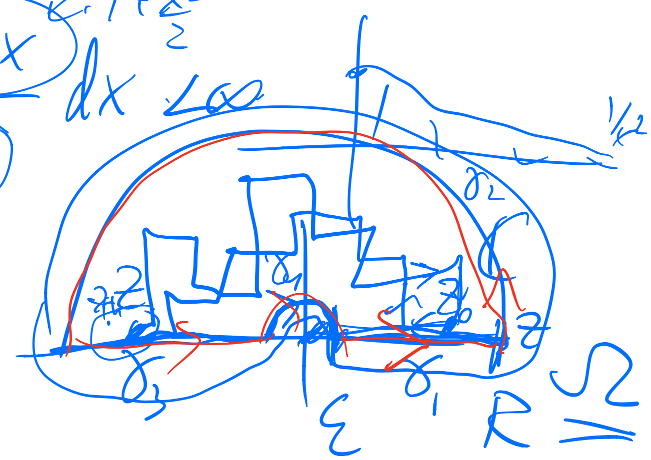
Cauchy's Thm in Disk: $f : \Omega \to \mathbb{C}$ holo, $\Omega \supset D$

$C = \partial D \Rightarrow f$ has a primitive in D. $\Rightarrow \oint_\gamma f(z)dz = 0$, $\gamma \subset D$



$$\int_{z_0}^{z} f(w)\,dw = F(z).$$

Ex 2: $I = \int_0^\infty \frac{1-\cos x}{x^2}\,dx < \infty$ $\quad \leq 1 + \frac{x^2}{2}$



$$f(z) = \frac{1-e^{iz}}{z^2}.$$

Fix $R>0$, $\varepsilon>0$, $\quad \gamma = \gamma_1 + \dots + \gamma_4$.

Consider $\oint_\gamma f(z)dz = 0$ ✓ (claim). $\quad f$ holo on Ω.

Claim: f has Primitive in Ω. ✓ $F(z) = \int f(w)dw$.

$\int_\Delta f = 0 \Rightarrow \int f = 0$, $\to$ well-defined? $\to$ rectilinear path $z_0 \to z$

Δ Goursat. rectangles ✓

$$\int_{\gamma_1} = \int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2}\cdot 1\cdot dx \xrightarrow[\varepsilon\to 0]{R\to\infty} (\mathrm{Re}) = I.$$

$\gamma_1 \Leftarrow \underline{z = x}, \ \varepsilon < x < R, \ z' = 1$

$$\int_{\gamma_3} = \int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2}\cdot 1\cdot dx = \int_{\varepsilon}^{R} \frac{1-e^{-ix}}{x^2}\,dx$$

$\underline{x \mapsto -x}$

$\gamma_3 \Leftarrow z = x, \ -R < x < -\varepsilon, \ z' = 1$

$$\left|\int_{\gamma_2} = \int_0^{\pi} \frac{1-e^{iRe^{i\theta}}}{R^2 e^{2i\theta}}\cdot Re^{i\theta}\cdot i\,d\theta\right|$$

$e^{iR\cos\theta}$, $|\cdot| = 1$, $\dfrac{e^{iR\sin\theta}}{e^{-R\sin\theta}}$, ≥ 0

$\gamma_2 \Leftarrow z = Re^{i\theta}, \ 0 < \theta < \pi, \ z' = Re^{i\theta}\cdot i$

$$\leq \int_0^{\pi} \frac{1+\overbrace{e^{-R\sin\theta}}^{\leq 1}}{R^2}\cdot R\,d\theta \leq 2\pi/R \to 0.$$

$$\int_{\gamma_4} = -\int_0^{\pi} \frac{1-e^{iz}}{z^2}\,dz$$

$\gamma_4 \Leftarrow z = \underline{\varepsilon e^{i\theta}}, \ \underline{\theta : \pi \to 0}, \ z' = \varepsilon e^{i\theta}\cdot i$

for $|z| = \varepsilon \to 0$, $e^{iz} = 1 + iz + \underbrace{o_{z\to 0}(|z|)}_{E}$.

$$\boxed{\frac{E}{|z|} \to 0 \text{ as } z \to 0}$$

$$\frac{1-e^{iz}}{z^2} = \frac{-iz + o_{z\to 0}(|z|)}{z^2}$$

$$\frac{1}{|z|}$$

$$= \int_{\gamma_4} \left[\frac{-i}{z} + o_{z\to 0}\left(\frac{1}{|z|}\right)\right] dz$$

$$o_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\right)$$

$$= -\int_0^{\pi} \frac{-i}{\cancel{\varepsilon e^{i\theta}}} \cdot \cancel{\varepsilon e^{i\theta}} \cdot i\, d\theta + o_{\varepsilon\to 0}\left(\int_0^{\pi} \frac{\varepsilon\, d\theta}{\varepsilon}\right),$$

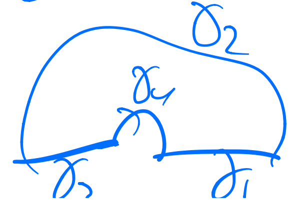
$$\left|\int_{\gamma_4} \frac{E}{|z|^2}\, dz\right| \le \int_0^{\pi} \frac{|E|}{\varepsilon^2} \cdot \varepsilon \cdot d\theta = \pi \cdot \frac{|E|}{\varepsilon} \to 0,$$

$z \to 0 \Leftrightarrow \varepsilon \to 0$.

$$= -\pi + o_{\varepsilon\to 0}(1) \to -\pi$$

as $\varepsilon \to 0$.

$$0 = \oint_{\gamma} f = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4}$$



$$\underbrace{0 \qquad\qquad \to -\pi}_{Re(\cdot) = \Sigma.}$$

$$\boxed{\frac{1}{z-A}}$$

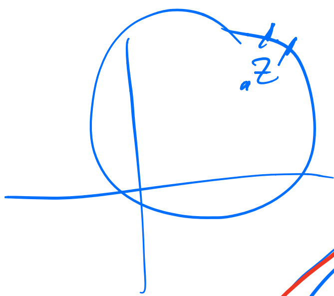
As $R \to \infty$ & $\varepsilon \to 0$ & take $Re[\cdot]$,

$$0 = 2\pi - \pi, \quad \int_0^\infty \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}.$$

<u>Thm</u> <u>Cauchy Integral Representation Formula</u>: Let $f: \Omega \to \mathbb{C}$, $\Omega \supseteq \bar{D}$, hol'c, $C = \partial D$. Then $\forall z \in D$,

$$\boxed{\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw = \underline{f(z)}.}$$

<u>Pf</u>: Let $g(w) = \frac{f(w)}{w-z}$ <u>not</u> hol'c on D. (Fix $z \in D$.)



D instead of $\int_C \to \int_\gamma$ "toy contour".

Fix $\varepsilon > 0$, $\eta > 0$, "Keyhole contour". Let $\gamma = \gamma_{\varepsilon,\eta}$.

$\Omega_1 \to$

g is hol'c on Ω_1 $\Rightarrow$ g has primitive, $\Rightarrow$

$$0 = \int_\gamma g = \int_{\gamma_1} + \cancel{\int_{\gamma_2}} + \int_{\gamma_3} + \cancel{\int_{\gamma_4}} \Rightarrow I - i\, 2\pi \cdot f(z)$$

$\downarrow \eta \to 0$

$$\int_C f = I$$

as $\eta \to 0$ (with $\varepsilon > 0$ fixed), these cancel. by cont of g away from z,

As $\eta \to 0$, $\int_{\gamma_3} \to -\int_0^{2\pi} \frac{f(z) + o_{w\to z}(1)}{w - z}\, dw$

$w = z + \varepsilon e^{i\theta}$, $w' = \varepsilon e^{i\theta} i$. Near z,

$f(w) = f(z) + o_{w \to z}(1)$. (only using f is cont!)

$$= -f(z) \cdot \int_0^{2\pi} \frac{1}{\varepsilon e^{i\theta}}\, \varepsilon e^{i\theta} i \, d\theta \; - \int_0^{2\pi} \frac{o_{w\to z}(1)}{|w - z|}\, dw$$

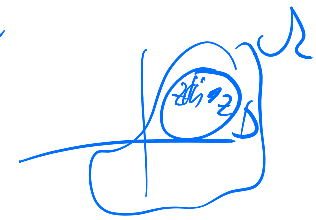
$$= -i f(z) \cdot 2\pi.$$

$o_{\varepsilon\to 0}\left(\frac{1}{\varepsilon} \cdot \varepsilon\right) \to 0$ as $\varepsilon \to 0$. $\ell(\gamma_2)$.

Cor: f holo $\Longrightarrow C^\infty$.

$f: \Omega \to \mathbb{C}$ holo near z, $\exists D \ni z$, $C = \partial D$,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}\,dw.$$



h small s.t. $z+h \in D$.

Pf: $n=0$ ✓ Assume $n=k-1$.

Look at $\frac{1}{h}\left(f^{(k-1)}(z+h) - f^{(k-1)}(z)\right)$

$$= \frac{1}{h}\frac{(k-1)!}{2\pi i}\int_C f(w)\left[\frac{1}{(w-(z+h))^k} - \frac{1}{(w-z)^k}\right]dw.$$

$$A^k - B^k = (A-B)(A^{k-1} + A^{k-2}B + \dots + B^{k-1})$$

$$\left(\frac{1}{w-(z+h)} - \frac{1}{w-z}\right);$$

$$\left(\frac{1}{(w-(z+h))^{k-1}} + \dots + \frac{1}{(w-z)^{k-1}}\right).$$

$$\frac{(w-z)-(w-(z+h))}{(w-(z+h))(w-z)} = \frac{h}{(w-z)(w-(z+h))}.$$

Take $I_n = \dfrac{(k+1)!}{2\pi i} \displaystyle\int_C f(w) \cdot \frac{k}{(w-z)^2 \cdot (w-z)^{k+1}}\, dw$.

$n \geq 0$,

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abs convergence, $|w-z| > \varepsilon > 0$

$|w-(z+h)| > \varepsilon > 0$,
