Last time: $z:[a,b]\to\mathbb{C}$ "smooth (piecewise) $\hat{C}^1$ parametric curve",

$\tilde{z}:[c,d]\to\mathbb{C}$ is equiv to z iff: $\exists\, t:[c,d]\to[a,b]$ bij, $t'>0$, s.t. $\tilde{z}(s)=z(t(s))$.

$\gamma =$ "curve" $= [z] := \{z/\sim\}$.

Def: $\displaystyle\int_\gamma f(z)\,dz = \int_\gamma f = \int_a^b f(z(t))\,z'(t)\,dt$.

Proved: If f is cont & has primitive F (F hol & $F'=f$) on $\Omega \supset \gamma$, $\gamma: w_0 \to w_1$,

$\Rightarrow \displaystyle\int_\gamma f = F(w_1) - F(w_0)$. Cor: γ closed $\displaystyle\oint_\gamma f = 0$.

Green's Thm: $\overline{R} \subset \Omega$ open, $\partial R = \gamma$. γ simple encloses R

& $P, Q : \mathbb{R}^2 \to \mathbb{R}$, cont diff'ble.

$$\int_\gamma P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Cauchy's Thm: If f holo on $\Omega \supset \overline{R}$ where $\boxed{\partial R = \gamma}$

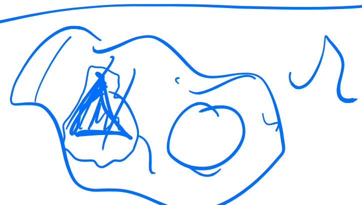
$\Rightarrow \oint_\gamma f = 0$. $\quad f = u + iv$, $z = x + iy$, $dz = dx + i\,dy$.

pf:
$$\int_\gamma f\,dz = \int_\gamma (u+iv)(dx + i\,dy) = \int_\gamma \overset{P}{\tilde{u}}\,dx - \overset{Q}{\tilde{v}}\,dy + i\left(\int_\gamma \overset{Q}{\tilde{u}}\,dy + \overset{P}{\tilde{v}}\,dx\right)$$

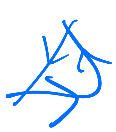
$$\underset{\text{(Green)}}{=} \iint_R (\underbrace{-v_x - u_y}_{=0})\,dx\,dy + i \iint_R (\underbrace{u_x - v_y}_{=0})\,dx\,dy \quad \Rightarrow 0.$$
$\quad$(C-R)

To avoid making γ precise via Jordan Curve Thm we bypass difficulties w/ beautiful argument of Goursat. (Cauchy for triangles).

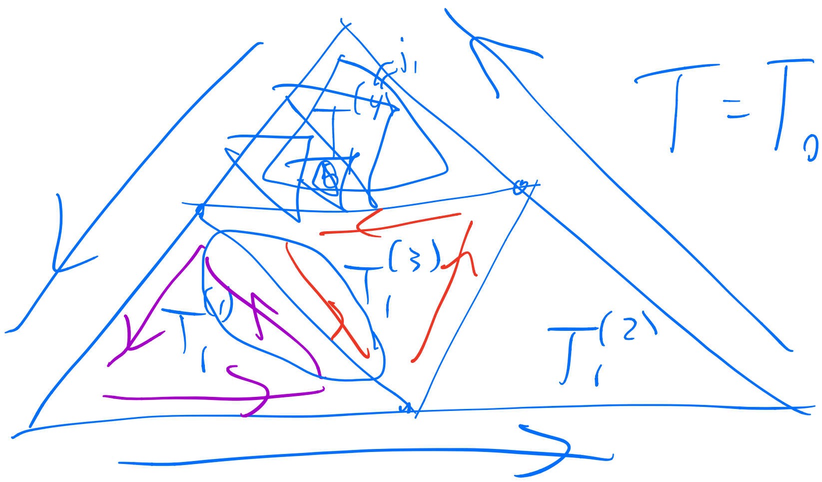
Thm (Goursat): Let $f : \Omega \to \mathbb{C}$, Ω open, $\partial T \subset T \subset \Omega$.



f holo $\Rightarrow \oint_{\partial T} f(z)\,dz = 0$.



Pf:



$T = T_0$.

Claim: $$\int_{\partial T_0} f = \underbrace{\int_{\partial T_1^{(1)}} f + \int_{\partial T_1^{(2)}} f + \cdots + \int_{\partial T_1^{(4)}} f}_{\text{interior contributions cancel.}}$$

$$\Rightarrow \left| \int_{\partial T_0} f \right| \le \left| \int_{\partial T_1^{(1)}} f \right| + \cdots + \left| \int_{\partial T_1^{(4)}} f \right|$$

Let $j_1 = \operatorname{argmax}\{\ \}$ = the j s.t. $\left|\int_{\partial T_1^{(j)}} f\right|$ largest.

$$\le 4 \left| \int_{\partial T_1^{(j_1)}} f \right|.$$

Zoom in on $T_1^{(j_1)}$. Proceed by induction:

$$\boxed{\left| \int_{\partial T_N^{(j_N)}} f \right| \le 4 \cdot \left| \int_{\partial T_{N+1}^{(j_{N+1})}} f \right|}$$

$T_0 \supset T_1 \supset T_2 \cdots \supset T_N \supset \cdots$ ← cpt

✓ Claim: If $T_N \supset T_{N+1}$ cpt sets (nonempty) & $\operatorname{diam}(T_N) \to 0$,

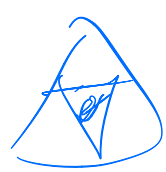
$\Rightarrow \exists!\, z$ s.t. $\bigcap_N T_N = \{z\}$.

Pf: let $w_N = \operatorname{midpt}(T_N)$

$\{w_N\} \subset T_0 \subset$ cpt $\Rightarrow$

$\exists\, z = \lim w_{N_j} \in T_N \quad \forall N.$

$\Rightarrow z \in \bigcap T_N.$



If $\exists\, w \in \bigcap T_N$,

$\Rightarrow \underbrace{d(z,w)} \leq \operatorname{diam} T_N \to 0.$

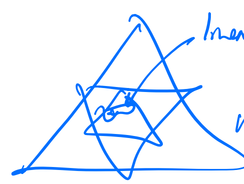
$\Rightarrow w = z.$

This $z \in T_0 \subset \Omega$,

& f holo in neigh of z,

Near z (i.e. $\underbrace{\text{for } N \text{ suff large}}$)

that is, $\forall\, w \in T_N$,



linearize f near z.

$$f(w) = \underbrace{f(z) + f'(z)(w-z)} + \underbrace{o_{w\to z}(|w-z|)},$$

$\leq \underset{N\to\infty}{O}(1) \cdot \operatorname{diam}(T_N)$.

$$\left| \int_{\partial T_N} f(w)\,dw \right| = \left| \int_{\partial T_N} \Big(\underbrace{f(z)\,dw + f'(z)(w-z)\,dw}_{\text{has obvious primitive} \Rightarrow \oint_{\partial T_N} = 0.} + o(|w-z|)\,dw \Big) \right|$$

$$= \left| \int_{\partial T_N} o(|w-z|)\,dw \right| \le \sup o \cdot \ell(\partial T_N)$$

$$\le o_N(1) \cdot \operatorname{diam}(T_N) \cdot \ell(\partial T_N)$$

$$\left| \int_{\partial T_0} f \right| \le 4 \left| \int_{\partial T_1} f \right| \le 4^N \cdot \left| \int_{\partial T_N} f \right|$$

$$\le 4^N\, o(1) \cdot 2^{-N} \cdot \operatorname{diam}(T_0) \cdot 2^{-N}\, \ell(\partial T_0)$$

$$\to 0. \quad \Rightarrow \int_{\partial T_0} f = 0.$$

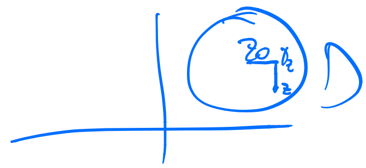
Prop'd: $f:\underset{\cup\!\!\!\,\top\ T}{\Omega} \to \mathbb{C}$ holo $\Rightarrow \int_{\partial T} f = 0$.

Cor: $f:\underset{\cup\ \text{rectangle } R}{\Omega} \to \mathbb{C}$ holo $\Rightarrow \int_{\partial R} f = 0$.

$\partial R = \partial T_1 + \partial T_2$

Thm (Cauchy): If f is holo on D open

$\Rightarrow f$ has a primitive.



pf: Let $z_0 = \text{center}(D)$, $\forall z \in D$, define

$\gamma_z =$ from z_0 move horiz, then vertically

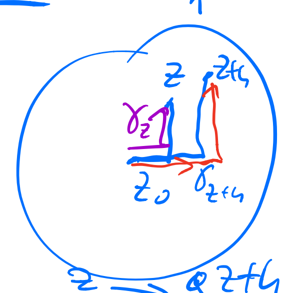
Define $F(z) := \int_{\gamma_z} f(w)\,dw$ **Claim:** $\dfrac{F(z+h)-F(z)}{h} \to f(z)$ $\forall z \in D$.

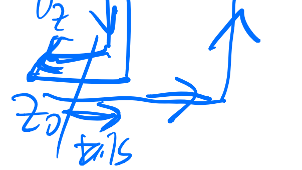
Make h small s.t. $z+h \in D$.

Look at $\frac{1}{h}\left(\underbrace{F(z+h)} - F(z)\right)$

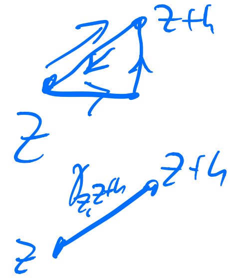
$= \frac{1}{h}\left(\int_{\gamma_{z+h}} f - \underbrace{\int_{\gamma_z} f}\right) = \frac{1}{h}$

$+ \int_{\gamma_z^-} f$





tails cancel. make rectangle. $\int_{\partial R} f = 0$.



make triangle. $\int_{\partial T} f = 0$.

So $\frac{1}{h}\big(F(z+h) - F(z)\big) = \underbrace{\frac{1}{h}\int_{\gamma_{z,z+h}} f(w)\,dw}$.

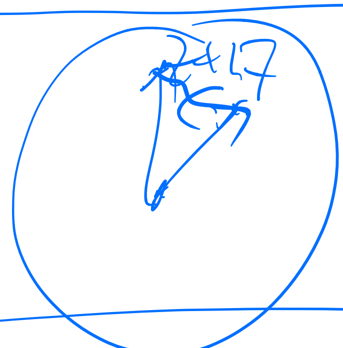
All we use now is $f = \underline{\text{cont}}$.

Cont $\Rightarrow$ $f(w) = f(z) + o_{w\to z}(1)$.

rate doesn't matter.

$$= \frac{1}{h}\Big[\underbrace{\int_\gamma \underbrace{f(z)}_{\text{constant}}\,dw}_{f(z)\cdot w\big|_z^{z+h}} + \underbrace{\int_\gamma o_{w\to z}(1)\,dw}_{+\,O(\ell(\gamma)\cdot o_{h\to 0}(1))}\Big].$$

$$= \frac{1}{h}\Big[f(z)\cdot h + O(|h|\cdot o(1))\Big] = \underbrace{f(z) + o(1)}_{F'(z)}.$$



better!

$X = O(Y) \iff \exists C$ s.t. $X \le CY$.

$X = o(Y) \iff \frac{X}{Y} \to 0$.

Ex 4: Fourier transform of Gaussian? $f(x) = e^{-\pi x^2}$.

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi}\, dx.$$

$$= \int_{\mathbb{R}} \underbrace{e^{-\pi x^2}} \underbrace{e^{-2\pi i x \xi}}_{|\cdot| = 1} dx, \ |\cdot| < \infty.$$

$\hat{f}(0)$? $\hat{f}(0) = \int_{\mathbb{R}} e^{-\pi x^2} dx = 1$,

$I =$

$$I^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \overbrace{e^{-\pi x^2} e^{-\pi y^2}}^{e^{-\pi(x^2+y^2)}} dx\, dy$$

$$\text{(Polar Coords)} = \int_0^\infty \int_0^{2\pi} e^{-\pi r^2} \, r\,dr\,d\theta$$

$x = r\cos\theta, \; y = r\sin\theta$

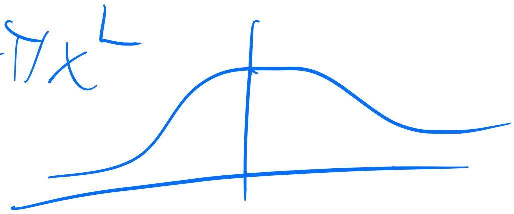
$$\frac{d}{dr}\left(\frac{e^{-\pi r^2}}{-2\pi}\right) = \frac{e^{-\pi r^2}}{-2\pi}\left(-2\pi r\right),$$

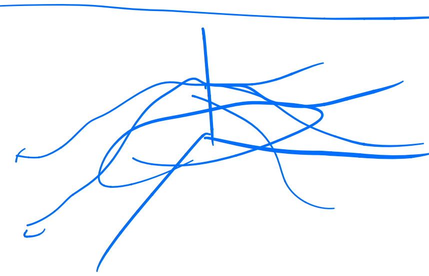
$$= 2\pi \cdot \left(\frac{e^{-\pi r^2}}{-2\pi}\right)\Bigg|_0^\infty = 2\pi\left(0 - \frac{1}{-2\pi}\right)$$

$$= 1,$$

$I^2 = 1, \; I = \pm 1, \; I > 0.$

$e^{-\pi x^2}$





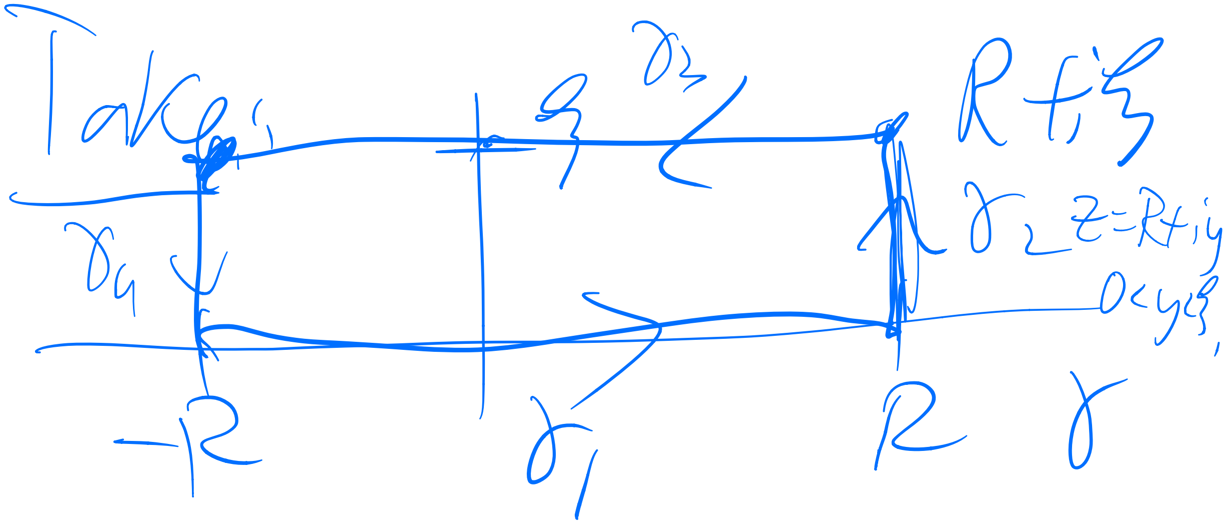
<u>Know:</u> $\hat{f}(0) = 1$. $\xi > 0$.

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x \xi} \, dx.$$

Look at $f(z) = e^{-\pi z^2}$.

entire (holo on $\Omega = \mathbb{C}$).

Take:



Fix R ($R \to \infty$).

$$\int_\gamma f\,dz = 0$$ holc, rectangle

$$\| \qquad z = x,\ -R<x<R,\ z'(t)=1.$$

$$\int_{\gamma_1} + \cdots + \int_{\gamma_4} . \qquad \int_{\gamma_1} f(z)\,dz$$

$$\int_{-R}^{R} e^{-\pi x^2} \cdot 1 \cdot dx \to 1$$

$$z = x+i\xi,\ -R<x<+R,\ z'=1,$$

$$-\int_{\gamma_3} = -\int_{-R}^{R} \underbrace{e^{-\pi(x+i\xi)^2}}_{e^{-\pi(x^2+2xi\xi-\xi^2)}} \cdot 1 \cdot dx.$$

$$= -e^{+\pi\xi^2} \cdot \underbrace{\int_{-R}^{R} e^{-\pi x^2} e^{-2\pi x\xi i}\,dx}_{\to \hat{f}(\xi).}$$

$$\left|\int_{\gamma_2}\right| = \left|\int_0^{\xi} e^{-\pi\underbrace{(R+iy)^2}_{R^2+2iy-y^2}} \cdot i\,dy\right| \le \underbrace{\int_0^{\xi} e^{\pi y^2} \cdot e^{-\pi R^2} \cdot |i| \cdot dy}$$

$\gamma_2 \leftarrow z = R+iy,\ 0<y<\xi,\ z'(y) = i$

$$\le \int_0^{\xi} e^{\pi\xi^2} e^{-\pi R^2}\,dy = \xi \cdot e^{\pi\xi^2} e^{-\pi R^2} \to 0$$

Same $\left|\int_{\gamma_4}\right| \to 0$. As $R \to \infty$

$$0 = \int_{\gamma} = \int_{\gamma_1} + \int_{\gamma_3} + 0 + 0$$

$$0 = 1 - e^{\pi\xi^2}\hat{f}(\xi),$$

$$\Rightarrow \hat{f}(\xi) = e^{-\pi\xi^2}.$$