Recall: Thm: F incr & cont $\Rightarrow$ F' exists a.e.

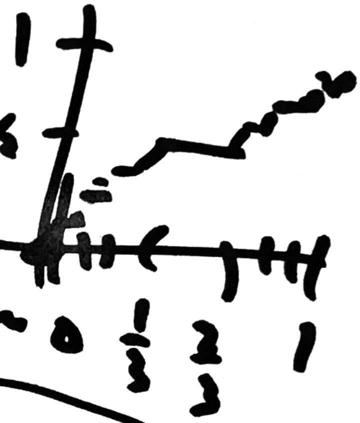
$(D^+ = D_+ = D^- = D_-$ a.e.$)$.

Cor: If F inc & cont & bdd on $[a,b]$

Then F' is meas'ble, non-neg &

$$\int_a^b F'(x)\,dx \le F(b) - F(a) \quad (\Rightarrow F' \in L^1).$$

Rmk!!!: $=$ is FALSE.



Non-e.g. Cantor-Lebesgue function $F_N \to F$

Pf: let $G_N(x) := \dfrac{F(x+1/N) - F(x)}{1/N}$ ← meas'ble,

$G_N \to F'$ a.e. (on $[a, b-\frac{1}{N}]$) $\Rightarrow$ on $[a,b]$).

$\Rightarrow F'$ is meas'ble & non-neg.

Fatou: $\displaystyle\int_a^b \underbrace{\liminf G_N}_{F'} \le \liminf \int_a^b G_N$

$$= \liminf \frac{1}{1/N}\left[\int_a^b F(x+\tfrac{1}{N})\,dx - \int_a^b F(x)\,dx\right]$$

$$= \liminf \frac{1}{1/n}\left[\int_b^{b+1/n} F(x)\,dx\right] - \frac{1}{1/n}\left[\int_a^{a+1/n} F(x)\,dx\right].$$

extend as constant on $[b, b+1)$.

$\to F(b)$ $\qquad$ $\to F(a)$

<u>Recall:</u> Lebesgue Diff Thm:

$$\lim_{\substack{m(B)\to 0\\ x\in B}} \frac{1}{m(B)}\int_B f = f(x) \text{ a.e.} \qquad (f\in L^1_{loc}).$$

<u>Recall:</u> Proved: $f\in L^1 \Rightarrow \forall \varepsilon>0\ \exists\delta$ s.t. $\forall E\in\mathcal{M}$, $m(E)<\delta$, $\int_E |f| < \varepsilon$.

<u>Def:</u> F on $[a,b]$ is absolutely continuous <u>if:</u> $\forall\varepsilon>0\ \exists\delta>0$: $\forall$ $\bigsqcup_1^N (a_k,b_k)$ with $m\left(\bigsqcup_1^N (a_k,b_k)\right)<\delta$, $\sum_{k=1}^N |F(b_k)-F(a_k)| < \varepsilon$.

<u>Rmk:</u> Abs cont $\Rightarrow$ unif cont

<u>Exercise:</u> Why Cantor-Lebesgue not Abs Cont (from def)?

Target Thm: F is abs cont $\Rightarrow$

$$\int_a^x F'(y)\,dy = F(x) - F(a).$$

Def: A Vitali covering $\mathcal{B}$ of E FS is a collection of balls B s.t. $\forall x \in E$ & $\forall \eta > 0$, $\exists B \in \mathcal{B}$ with $m(B) < \eta$, & $x \in B$.

i.e. can zoom in on any pt of E with balls from $\mathcal{B}$.

Covering lemma: If $E \subset \mathbb{R}^d$ has $m(E) < \infty$ & $\mathcal{B}$ is a Vitali cover, then $\forall \delta > 0$ $\exists S \subset \mathcal{B}$ finite, disjoint & $m\left(\bigsqcup_{B \in S} B\right) \geq m(E) - \delta$ $(m(E) > \delta)$.

 E

Pf: $m(E) < \infty \Rightarrow \exists K_1$ cpt $\subset E$ with $m(K_1) \geq \cancel{m(E)-\delta}\ \delta$. $\mathcal{B}$ covers $K_1 \Rightarrow$ $\exists \mathcal{B}_1 \subset \mathcal{B}$ finite covering K_1.

By OLD lemma $\exists\ S_1 \subset \mathcal{B}$, disjoint ~~sets~~ balls with $m\Big(\bigsqcup_{B\in S_1} B\Big) \geq \frac{1}{3^d}\, m\Big(\bigcup_{B\in \mathcal{B}_1} B\Big)$.

If $m\Big(\overbrace{\bigsqcup_{B\in S_1} B}^{S_1}\Big) \geq m(E)-\delta$ then done.

OTHERwise, $m(S_1) < m(E)-\delta$

so $m\big(\underbrace{E \setminus \bar{S}_1}_{E_1}\big) > \delta$.

<u>Observe:</u> $\{B \in \mathcal{B} \mid B \cap S_1 = \emptyset\}$ is Vitali cover for E_1. <u>Iterate.</u>

$S_2 \subset \mathcal{B}$ finite, disjoint & disjoint from S_1.

~~[illegible]~~ $\geq \frac{1}{3^d}\, m(K_1) \geq \frac{1}{3^d}\big(\cancel{m(E)-\delta}\ \delta\big)$.

$\to$ Get $K_2 \subset E_1$ with $m(K_2) \geq \delta$.

$\exists$ finite $\mathcal{B}_2 \supset S_2$ disjoint

Halt if $m(E \setminus (\bar{S}_1 \sqcup \bar{S}_2 \sqcup \cdots \sqcup \bar{S}_k)) \le \delta$

($\Rightarrow m(S_1 \sqcup \cdots \sqcup S_k) \ge m(E) - \delta$.).

Each new k adds $\ge \frac{\delta}{3^d}$ mass to

So Alg halts after finite time.

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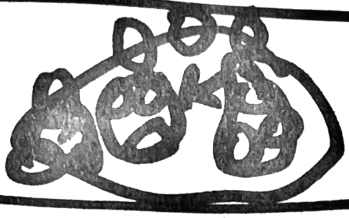
i.e.  E

<u>Cor:</u> Balls can be arranged s.t. $m(E \setminus S) < 2\delta$.

<u>pf.</u> Let $O \supset E$ be open s.t. $m(O) \le m(E) + \delta$.

Replace $\mathcal{B}$ by $\{B \in \mathcal{B} \mid B \subset O\}$.

So $m(E \setminus S) \le m(O) - m(\sqcup B)$

$$\le m(E) + \delta - (m(E) - \delta) = 2\delta.$$

Thm: F abs cont & $F' = 0$ a.e.
$\Rightarrow F \equiv$ constant.

Pf: Let $E = \{a < x < b \mid F' = 0\}$.

$\Rightarrow m(E) = b - a < \infty$. $\forall x \in E$, ~~$\lim_{h \to 0} F(\ldots)$~~

$$\lim_{h \to 0} \left| \frac{F(x+h) - F(x)}{h} \right| = 0.$$

Fix $\varepsilon > 0$. $\exists \delta > 0$ s.t. $\left(\forall \sum_k^m |(a_k, b_k)| < \delta \Rightarrow \sum |F(b_k) - F(a_k)| < \varepsilon / \right.$

$\forall x \in E$, $\forall$ small $h > 0$, $\exists$ open $x \in (a_x, b_x) \subset [a, b]$

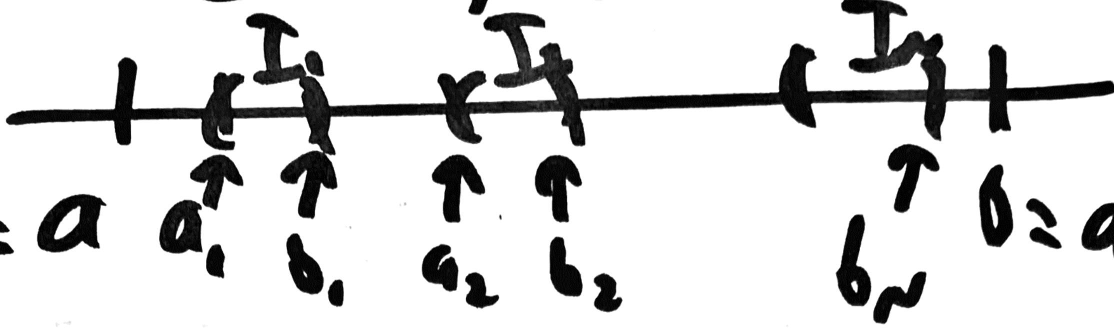
s.t. $b_x - a_x < h$ & $|F(b_x) - F(a_x)| \leq \varepsilon (b_x - a_x)$

Claim: These intervals are Vitali cover of E.

$(F(b_x) - F(x)) - (F(a_x) - F(x)) = F(b_x) - F(a_x)$.

So Vitali lemma $\Rightarrow \exists$ finite collection of disjoint intervals $I_1, \ldots, I_N = (a_N, b_N)$.

s.t. $\sum m(I_j) \geq m(E) - \delta = b - a - \delta.$



Complement $[a,b] \setminus \bigcup I_j = \bigsqcup [b_j, a_{j+1}]$

$m(\text{Complement}) \leq \delta$ so

$$\sum |F(b_j) - F(a_{j+1})| < \varepsilon.$$

So

$$|F(b) - F(b_N) + F(b_N) - F(a_N) + \dots - F(a)|$$

$$\leq \underbrace{\sum_{I_j} |F(b_j) - F(a_j)|}_{\leq \varepsilon \underbrace{\sum_{I_j} m(I_j)}_{\leq b-a}} + \underbrace{\sum_{J_j} |F(a_{j+1}) - F(b_j)|}_{< \varepsilon}.$$

$$\leq \varepsilon(b - a + 1).$$

$$\to 0.$$

Exercise: F abs cont $\Rightarrow T_F(x)$ is & P_F & N_F are.

Each P_F & N_F are cont $\Rightarrow F$ is.

Fund Thm of Lebesgue Integration:

If F is abs cont on $[a,b]$ then F' exists a.e., is integrable, &

$$\int_a^x F'(y)\,dy = F(x) - F(a) \quad \forall x.$$

Conversely, if $f \in L^1$ then $\exists F$ abs cont & $f = F'$ a.e., namely $F(x) = \int_a^x f(y)\,dy$.

Pf: F abs cont $\Rightarrow F'$ exists. Let

$$G(x) := \int_a^x F'(y)\,dy.$$

Then $G - F$ has 0 derivative $\Rightarrow$ const.