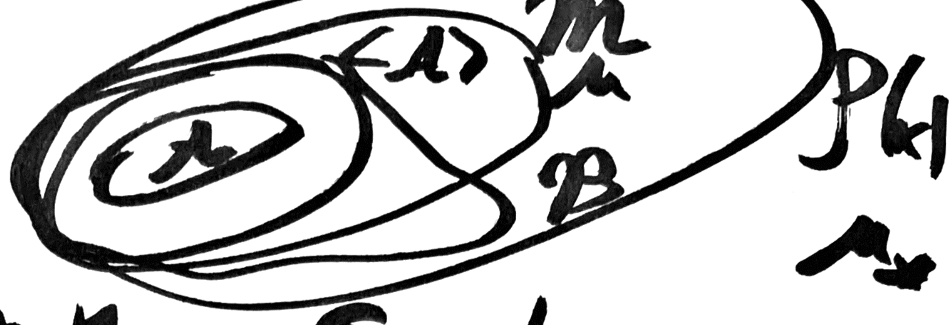
Correction:



If $\mathcal{B} \subset \langle \mathcal{A} \rangle$ is another σ-alg & ν meas on $\mathcal{B}$ s.t. $\nu|_{\mathcal{A}} = \mu_0$

$\Rightarrow \mathcal{B} \subseteq \mathfrak{M}$.

$$\mu^*(E) = \inf_{\substack{E \subset F \\ F \in \mathcal{A}_\sigma}} \mu^*(F) = \nu(F).$$

Issue 1 $\Rightarrow \nu = \mu^*$!!!

$\cap F = E$. $F \in \mathcal{A}_\delta$

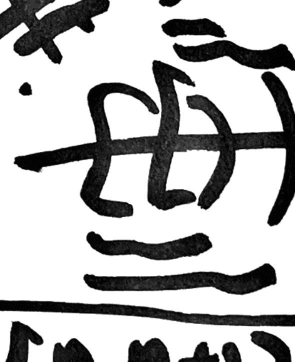
(If $\mathcal{B} = \mathfrak{M}$, $\Rightarrow \nu = \mu$).

If $\mathcal{B} \ni E \subset F \Rightarrow \exists F \in \mathcal{A}_\delta \supseteq E$.

Issue 2: if $\nu(E) < \infty \not\Rightarrow \nu(F) < \infty$.

Counterexample: $\mathcal{A}$ = finite unions of $[a,b) \subset \mathbb{R}$. $\mu_0([a,b)) = \infty$. $\cancel{\neq} \emptyset$.

$$\Rightarrow \mu^* = \mu = \begin{cases} \infty & \text{oth.} \\ 0 & \emptyset \end{cases}$$



$\nu(E) = \# E$.

$\mathcal{A} = [n, n+1)$. ν = Lebesgue.

Thm: let μ_0 σ-finite on $\mathcal{A}$ be premeasure, μ_* = exterior measure, $\mu = \mu_*|_{\mathcal{M}}$ measure from Caratheodory. let ν on $\mathcal{B}$ (σ-algebra) satisfy:

(0) $\nu|_{\mathcal{A}} = \mu_0$

(1) $\forall E \in \mathcal{B}$ $\exists F_n \in \mathcal{A}_\sigma$ s.t. $\bigcap F_n = E$.

(2) If $\nu(E) < \infty \Rightarrow \exists F \in \mathcal{A}_\sigma$ ($\supseteq E$): $\nu(F) < \infty$.

Then $\mathcal{B} \subseteq \mathcal{M}$ (& $\nu = \mu|_{\mathcal{B}}$).

Exercise: (2) $\Leftrightarrow$ σ-finite (?).

Recall: Proved Reisz Rep Thm:

Let $\mathcal{H} = L^2(\rho)$ & ℓ cont($\Leftrightarrow$ bdd) linear functional. $\exists! g \in \mathcal{H}$ s.t. $\forall f \in \mathcal{H}$,

$\ell(f) = (f, g)$.

($\Rightarrow \|\ell\| = \|g\|$)

(2)

<u>Recall pf:</u> $S = \{ f \mid \ell(f) = 0 \}$.

Then $S^\perp$ is 0 or <u>1-dim'l</u>.

If $h_1, h_2 \in S^\perp \Rightarrow \cancel{\ell(h_2)} h_1 - \frac{\ell(h_1)}{\ell(h_2)} h_2 \in S$.

Let $g = \frac{\overline{\ell(h)} \cdot h}{\|h\|^2}$ ✓.

<u>Thm Radon-Nikodym:</u> $(X, \mathcal{M})$ ~~[illegible]~~ meas'bl space, ν σ-finite signed measure; $\mu \geq 0$ σ-finite. Then $\exists!\ \nu_a, \nu_s$ s.t. $\nu = \nu_a + \nu_s$, $\nu_s \perp \mu$, $\nu_a \ll \mu$ & $\exists!\ f \in L^1_{\pm}(\mu)$ s.t. $\nu_a(E) = \int_E f \, d\mu$.

$\uparrow \frac{d\nu_a}{d\mu}$

<u>Pf:</u> Assume $\nu \geq 0$ & $\nu(X), \mu(X) < \infty$.

Let $\rho = \nu + \mu \Rightarrow \rho(X) < \infty$

For $\psi \in L^2(\rho)$ let

$$\boxed{\ell(\psi) := \int_X \psi \, d\nu}.$$ A linear functional.

Is ℓ bdd?

$$\left|\int_X \psi \cdot 1 \cdot d\nu\right| \le \int_X |\psi| \, d|\nu|$$

$$\overset{(C\text{-}S)}{\le} \left(\int_X |\psi|^2 d\nu\right)^{1/2} \cdot \left(\int_X 1 \cdot d\nu\right)^{1/2},$$

$\|\ell\| < \nu(X)^{1/2} < \infty$.

$\Rightarrow$ (Reisz) $\exists!\ g \in L^2(\rho)$. s.t.

$$\int_X \psi \, d\nu = \ell(\psi) = (\psi, g) = \int_X \psi \bar{g} \, d\rho.$$

$$= \int_X \psi \bar{g} \, d\nu + \int \psi \bar{g} \, d\mu$$

Apply to $\psi = \chi_E$.

$$\nu(E) = \int_X \chi_E \, d\nu = \int_X \chi_E \bar{g} \, d\nu + \int \chi_E \bar{g} \, d\mu$$

If $\rho(E) < \infty$ then

or

(4)

$$0 \le \nu(E) = \ell(\chi_E) = \int \chi_E \, d\nu = \int_E \bar{g} \, d\rho \le \rho(E)$$

$$\Rightarrow \quad 0 \le \frac{\int_E \bar{g} \, d\rho}{\rho(E)} \le 1. \quad \forall E \in \mathfrak{m}$$

with $\rho(E) > 0$.

<u>Exercise:</u> $\Rightarrow$ $g \in \mathbb{R}$, & $0 \le g \le 1$ a.e. ρ

Summary: $\exists! \, g \in [0,1] \cap L^2(X, \rho)$, s.t.

$$\ell(\psi) = \int \psi \, d\nu = \int \psi g \, d\rho$$

$$= \int \psi g \, d\nu + \int \psi g \, d\mu.$$

$$\Rightarrow \quad (*) \quad \int \psi \underbrace{(1-g)}_{\ge 0} \, d\nu = \int \psi g \, d\mu. \quad \text{Let}$$

$$A = \{ x \in X \mid 0 \le g(x) < 1 \} \quad \&$$

Fix $g \in [g]$.

$$B = \{ g = 1 \}. \qquad A \sqcup B = X.$$

Let $\nu_a(E) := \underset{(5)}{\nu}(E \cap A)$ &

Let $\nu_s(E) := \nu(E \cap B)$.

Claim: $\nu_s \perp \mu$. pf: $\nu_s(A) = 0$.

$\mu(B) = ?$ Let $\psi = \chi_B \Rightarrow$ (*) gives:

$$0 = \int_B (1-g)\,d\nu = \int_B g\,d\mu = \int_B 1\,d\mu = \mu(B).$$

Claim: $\nu_a \ll \mu$. pf: If $\mu(E) = 0$

$\Rightarrow$ Let $\psi = \chi_E$, (*) gives: $\int_{A \cap E} \underbrace{(1-g)}_{>0}\,d\nu = 0 = \int_E g\,d\mu$

$$0 = \int_E \underbrace{(1-g)}_{>0}\,d\nu_a \Rightarrow \nu_a(E) = 0.$$

Need Radon-Nikodym deriv:

Take $\psi = \chi_E(1+g+g^2+\dots+g^n)$ in (*)

$$\int_E \underbrace{(1+g+\dots+g^n)}_{=\frac{1-g^{n+1}}{1-g}}(1-g)\,d\nu_a = \int_E \underbrace{(1+g+\dots+g^n)\,g}_{\text{Monotone Conv}}\,d\mu$$

Dom

abs bdd by 1. Conv Thm $\Rightarrow$

$$\Rightarrow \underbrace{\int_E \frac{1}{1+g}(1-g)\, d\nu_a}_{\nu_a(E)} = \int_E \underbrace{\frac{g}{1+g}}_{\text{"}f \in L^1_+(\mu)\text{"}} d\mu.$$

If σ-finite, $\cup 2 0$ $X = \bigcup E_i = \bigcup S_j$

on S_i, $\nu_i = \nu_{i,a} + \nu_{i,s}$, $\nu_{i,a}(E) = \int_E f_i \, d\mu_i$

$\Rightarrow \nu_a = \sum \nu_{i,a}$, $\nu_s = \sum \nu_{i,s}$, $f = \sum f_i$

If ν signed, $\nu = \nu^+ - \nu^-$.

Unique? $\nu = \nu_a + \nu_s = \nu_a' + \nu_s'$

$\Rightarrow$ $\nu_a - \nu_a' \ll \mu$

$\quad\quad \shortparallel \quad\quad \Rightarrow \nu_a - \nu_a' = 0$

$\nu_s' - \nu_s \perp \mu$. $\quad = \nu_s - \nu_s'$.

If $\int_E f\, d\mu = \int_E f'\, d\mu \Rightarrow f = f'$.
$\forall E$ ...

(7)