Recall: ν signed measure on $(X, \mathcal{M})$ if $\nu : \mathcal{M} \to [-\infty, \infty]$ only one.

Proved: Hahn Decomp: $X = P \sqcup N$

$\forall E \subset P$, $\nu(E) \geq 0$, same N. & "$P' \sqcup N'$", then $\nu(P \Delta P') = 0$. $= \nu(N \Delta N')$

Def: ν is supported on $A \subset X$ if: $\forall E \in \mathcal{M}$, $\nu(E) = \nu(E \cap A)$.

Def: $\nu_1 \perp \nu_2$ mutually singular if $\exists A_1 \cap A_2 = \emptyset$ with ν_i supp on A_i.

Equiv, $\exists A_1 \sqcup A_2 = X$ s.t. A_i^c null for ν_i

Jordan Decomp Thm: Given ν $\exists!$ pos measures $\nu^+ \perp \nu^-$ s.t. $\nu = \nu^+ - \nu^-$.

pf: Let $X = P \sqcup N$ be a Hahn decomp, $\nu^+(E) = \nu(E \cap P)$ & $\nu^-(E) = -\nu(E \cap N)$.
≥ 0.

If $\nu = \mu^+ - \mu^-$ is another w/ $\mu^+ \perp \mu^-$.

$\Rightarrow \exists\ P' \sqcup N' = X$ s.t. $\mu^+(N') = 0 = \mu^-(P')$

$\Rightarrow P \Delta P'$ null for ν, $\Rightarrow \forall\ E \in \mathcal{M}$,

$\mu^+(E) = \mu^+(E \cap P') = \nu(E \cap P')$

$= \nu(E \cap P) = \nu^+(E)$. (Same $\mu^- = \nu^-$)

<u>Def.</u> $|\nu|$ = <u>total variation</u> $= \nu^+ + \nu^-$.

pos var, neg var

<u>Exercises</u>: ① $E \in \mathcal{M}$, is ν-null $\Leftrightarrow |\nu|(E) = 0$.

② $\nu \perp \mu \Leftrightarrow |\nu| \perp \mu \Leftrightarrow (\nu^+ \perp \mu\ \&\ \nu^- \perp \mu)$

<u>Rmk</u>: ① $|\nu| < \infty \Rightarrow \nu^+(X), \nu^-(X) < \infty$.

② $\nu(E) = \int_E (\chi_P - \chi_N)\, d|\nu|$.

<u>Def.</u> $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$. &

$\int f d\nu = \int f d\nu^+ - \int f d\nu^-$.

(2)

ν is σ-finite $\Leftrightarrow$ $|\nu|$ is.

Def.: If ν signed meas & $\mu \geq 0$. we say $\nu \ll \mu$ (absolutely cont). if $\mu(E)=0 \Rightarrow \nu(E)=0$.

Recall $f \in L^1_{\pm}(\mu)$ (extended integrable) if $f^{\pm} \in L^1(\mu)$ at least one of.

Thm (Lebesgue-Radon-Nikodym): Let ν be σ-finite, $\mu \geq 0$, σ-finite & assume $\nu \ll \mu$. Then $\exists!\, [f] \in L^1_{\pm}(\mu)$ s.t. $\nu(E) = \int_E f \, d\mu \quad \forall E \in \mathcal{m}$.

$\boxed{X \gg 0}$

In general (not assuming $\nu \ll \mu$), $\exists!\ \nu = \nu_a + \nu_s$ s.t. $\nu_s \perp \mu$ & $\nu_a \ll \mu$.

$\rightarrow d\nu = f\, d\mu$, $f = \frac{d\nu}{d\mu}$ = "Radon-Nikodym derivative".

"pf": (von Neumann) Assume $\nu \geq 0$ & $\nu(X), \mu(X) < \infty$.

Let $\rho := \nu + \mu$. finite measure on X.

Consider Hilbert space $L^2(\rho)$

$$= \left\{ [f] : X \to \mathbb{C} \;\middle|\; \underbrace{\left[\int |f|^2 d\rho\right]^{1/2}}^{\|f\|_2} < \infty \right\}.$$

Unlike L^1, L^2 has a sesquilinear product

$f, g \in L^2$, $(f,g) = \int f \cdot \bar{g} \, d\rho$. $\uparrow$ $3/2$.

Prop: (i) $L^2(X, \rho)$ is vector space ✓

(ii) $\|f\|_2 \geq 0$ & $\|f\|_2 = 0 \Rightarrow [f] = 0$. ✓

(iii) Δ-ineq $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$.

(iv) (Cauchy-Schwarz) $f, g \in L^2 \Rightarrow |f \cdot g| \in L^1$

conseq of Hölder $\int |f g| d\rho \leq \|f\|_2 \cdot \|g\|_2$.

(v) $(f_1 + f_2, g) = (f_1, g) + (f_2, g)$, $(cf, g) = c(f,g)$

$(f, g_1 + g_2) = (f, g_1) + (f, g_2)$, $(f, cg) = (f,g)\bar{c}$.

pf: $|f+g| \le 2\max(|f|, |g|)$

$\Rightarrow |f+g|^2 \le 4(|f|^2+|g|^2) \Rightarrow$ (i).

pf (ii): If $\|f\|=0$ ✓. Assume $\|f\|=1 = \|g\|$.

$(A-B)^2 \ge 0 \Rightarrow 2AB \le A^2+B^2$

$$\int |f\cdot g| \le \int \tfrac{1}{2}(|f|^2+|g|^2) = 1.$$

In general, set $\tilde{f} = \frac{f}{\|f\|}$, $\tilde{g} = \frac{g}{\|g\|}$.

$$\int |\tilde{f}\cdot\tilde{g}| \le 1 \Rightarrow \int |f\cdot g| \le \|f\|\cdot\|g\|.$$

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pf (iii): $\|f+g\|^2 = (f+g, f+g) \le \|f\|^2+\|g\|^2 + 2|(f,g)|$.

$$\le \underbrace{\|f\|^2+\|g\|^2+2\|f\|\cdot\|g\|}_{(\|f\|+\|g\|)^2}.$$

Thm (Reisz-Fischer): L^2 is complete.

pf: same. (Exercise) ∎

$f_n \to f$ in L^2, f_n
$\|f_n - f\| \to 0$.

Rmk: Given $g \in L^2(\rho) = \mathcal{H}$. Define

$\ell_g = \ell : \mathcal{H} \to \mathbb{C}$ linear functional:

$$\ell(f) = (f, g).$$

Rmk: ℓ is **continuous** i.e. if $f_n \xrightarrow{L^2} f \Rightarrow \ell(f_n) \to \ell(f)$

pf: $|\ell(f_n) - \ell(f)| = |(f_n - f, g)| \le \|g\| \cdot \|f_n - f\| \to 0.$

Rmk: ℓ is **bdd**, i.e.

operator norm $\displaystyle \|\ell\| = \sup_{\substack{f \in \mathcal{H} \\ \|f\| \neq 0}} \frac{|\ell(f)|}{\|f\|} < \infty.$

Lemma: $\|\ell\| = \|g\|.$

pf: $|\ell(f)| = |(f,g)| \le \|f\| \cdot \|g\|$

$\Rightarrow \|\ell\| \le \|g\|$ & $|\ell(g)| = \|g\|^2.$

Reisz Rep'n Thm: If ℓ cont linear functional on $\mathcal{H}$ then $\exists!\, g \in \mathcal{H}$ s.t.

$\ell(f) = (f, g)$ $\forall f \in \mathcal{H}.$ $\left(\Rightarrow \|\ell\| = \|g\| \right)$

Rmk: Bdd $\Rightarrow$ continuous:

Pf: $|\ell(f_n) - \ell(f)| = |\ell(f_n - f)|$

$$\leq \underbrace{\|\ell\|}_{<\infty} \cdot \underbrace{\|f_n - f\|}_{\to 0} \to 0$$

$\square$