Review: $L_f = \{ \lim \frac{1}{m(B)} \int_B |f(y) - f(x)| dy = 0 \}$

$g_x(y) = |f(y) - f(x)|$.

Ae x: Doesn't work, no uniformity in x!

New topic: **Def:** $(X, \mathcal{M})$ measurable space. ν is a signed measure if $\nu : \mathcal{M} \to [-\infty, \infty]$, $\nu(\emptyset) = 0$, countably additive: $\nu = \infty \Rightarrow \nu \neq -\infty$

only takes at most one of $\pm\infty$.

$\bigcup E_n = E$ (all sets in $\mathcal{M}$). $\Rightarrow$

$\nu(E) = \sum_n \nu(E_n)$. & if $|\nu(E)| < \infty$,

then converges absolutely.

disjoint.

Usual measures are "positive measures."

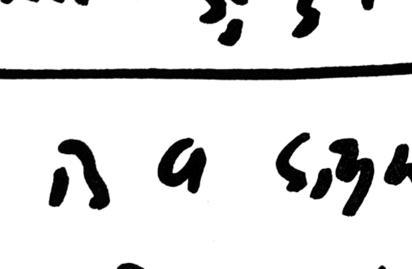
E.g.: $\nu(E) = \int_E f \, d\mu$ if $f \in L^1$.

Def: $f \in L^1_{\pm}$ if $f = f^+ - f^-$ & at least one of $f^{\pm} \in L^1$.

f is extended μ integrable.

Prop: ① $E_j \nearrow E$ then $\nu(E_j) \to \nu(E)$.

② $E_j \searrow E$ & $|\nu(E_1)| < \infty \Rightarrow \nu(E_j) \to \nu(E)$.

Pf: same  E_j $\quad S_j := E_j \setminus \bigcup^{j-1} E_k$.

& use disjoint S_j's + countable additivity.

Def: If ν is a signed measure then $E \in \mathcal{M}$ is called $\begin{cases} \text{positive (for } \nu) \\ \text{neg} \\ \text{null} \end{cases}$

if $\forall F \subset$ has $\nu(F) \geq 0$ / $\leq$

E.g.: $\nu(E) = \int_E f\,d\mu$, E is posit $f|_E \geq 0$.

Lemma: Countable union of $\left\{\begin{matrix}\text{pos}\\ \text{neg}\end{matrix}\right.$ sets is $\left\{\begin{matrix}\text{pos}\\ \text{neg}\end{matrix}\right.$.

Pf: Let E_n be positive, $E = \bigcup E_n$. Let $S_n = E_n \setminus \widetilde{\bigcup_{j=1}} E_j$. Then $S_n \subset E_n$, $E = \bigsqcup S_n$. Let $F \subset E$. **Want:** $\nu(F) \geq 0$.

$$\nu(F) = \sum \nu(\underbrace{F \cap S_n}_{\subseteq E_n}) \geq 0. \checkmark.$$

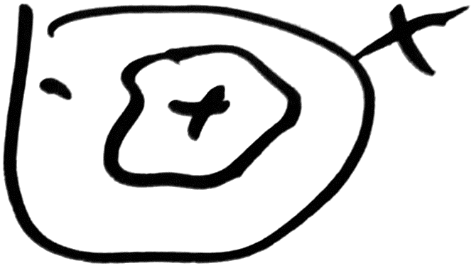
Hahn Decomp Thm: If ν signed measure $\Rightarrow \exists\, P$ positive, N neg $X = P \sqcup N$. & If P', N' another

(3)

decomposition, then

$$P \Delta P' = N \Delta N' \text{ are null}$$

<u>Lemma</u>: Assume $\nu \neq -\infty$. (or replace $\nu \to -\nu$).



If $D \in \mathcal{M}$, $\nu(D) \le 0$ then $\exists F \subset D$ negative & $\nu(F) \le \nu(D)$.

<u>pf</u>: let $D_0 = D$. let

$t_0 = \sup\{\nu(E) \mid E \subset D_0\}$. (could be ∞)

Since $\emptyset \subset D$, $t_0 \ge 0$. $\exists E_0 \subset D_0$ s.t. $\nu(E_0) \ge \min(1, t_0/2)$. let

$D_1 = D_0 \setminus E_0$. $t_1 = \sup\{\nu(E) \mid E \subset D_1\}$.

$D_n = D_{n-1} \setminus E_{n-1}$, $t_n = \cdots\cdots E \subset D_n\}$

$\exists\, E_n \subset D_n$ s.t. $\nu(E_n) \geq \min(1, t_n/2)$.

Note: E_n's disjoint, set ≤ 0.

$F = D \setminus \bigcup E_n$.

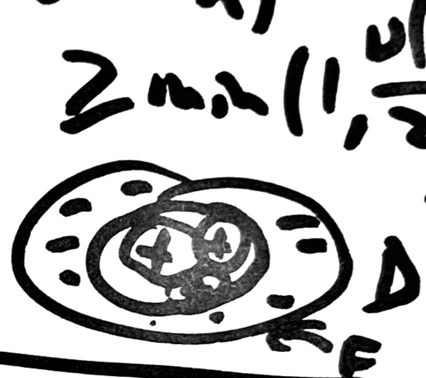
$\nu(F) = \nu(D) - \sum \nu(E_n) \leq \nu(D)$.

If $\exists\, S \subset F$ with $\nu(S) > 0$.

$\Rightarrow t_n \geq \nu(S)\ \forall n. \Rightarrow \nu(E_n) \geq \min\left(1, \frac{\nu(S)}{2}\right)$

$\Rightarrow \sum \nu(E_n) = \infty$

$\Rightarrow \nu(F) = -\infty$ ※.



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pf Hahn: let $N_0 = \emptyset$ let

$S_n = \inf\{\nu(E) \mid E \subset X \setminus N_n\} \leq 0$.

(Could be $-\infty$). $\neq -\infty$. So $\exists\, D_n \subset X \setminus N_n$

s.t. $\nu(D_n) \leq \max\left(\frac{S_n}{2}, -1\right) \leq 0$.

By Lemma, $\exists\, F_n \subset D_n$, $\nu(F_n) \le \nu(D_n)$ & F_n is negative. Let $N_{n+1} = N_n \cup F_n$

Let $N = \bigsqcup F_n = \bigcup N_n$. = neg.

Claim: $X \setminus N =: P$ is positive.

If not, $\exists\, E \subset P$ with $\nu(E) < 0$.

$\Rightarrow S_n \le \nu(E)$. $\Rightarrow \nu(F_n) \le \frac{\nu(E)}{2}$.

$\nu(N) = \sum \nu(F_n) \to -\infty$. ※.

$P \Delta P' = (P \cap N') \cup (P' \cap N) = N \Delta N'$

If P', N' another. are null
