Recall: $f \in L^1(\mathbb{R}^d)$,

B

$$f^*(x) := \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)|\,dy.$$ ← Hardy-Littlewood

Thm: (i) f^* measble (& cont) (ii) $f^* < \infty$ a.e.

(iii) $m(\{x / f^*(x) > \alpha\}) \leq \frac{3^d}{\alpha} \|f\|_{L^1}$.

Lebesgue Differentiation Thm: If $f \in L^1$, then a.e. x,

$$\lim_{\substack{m(B) \to 0 \\ x \in B}} \frac{1}{m(B)} \int_B f(y)\,dy = f(x). \quad (*)$$

Lemma: (i) simple functions are dense in L^1
(ii) step functions are (iii) $\overline{C(\mathbb{R}^d)} = L^1$.

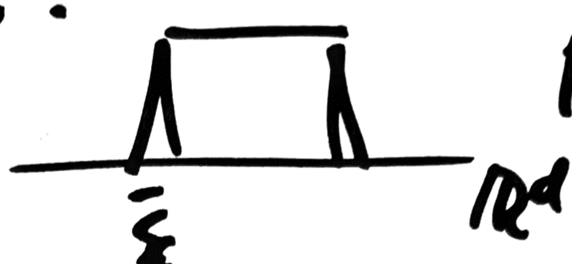
pf: (i) f meas $\Rightarrow \exists\, |\varphi_k| \leq |\varphi_{k+1}|$ s.t. $\varphi_k \to f$ p.w.

MCT/DCT $\|\varphi_k - f\|_{L^1} \to 0$.

(ii) ψ_k ~~simple~~ step $= \sum_{1}^{N} a_n \chi_{R_n}$. $\psi_k \to \varphi$.

(iii) $g \in C(\mathbb{R}^d)$ approximate R χ_R

$g \xrightarrow{L^1} \psi$.



For any $\alpha > 0$, let

$$E_\alpha = \left\{ x \,\middle|\, \overline{\lim_{\substack{x \in B \\ m(B) \to 0}}} \left| \frac{1}{m(B)} \int_B f(y)\,dy - f(x) \right| > \alpha \right\}.$$

$E = \{ x \mid (*) \text{ fails} \} \subset \bigcup_n E_{1/n}$. **Want:** $m(E_\alpha) = 0$

Fix $\varepsilon > 0$, $\exists\, g \in C(\mathbb{R}^d)$ s.t. $\|f - g\|_{L^1} < \varepsilon$.

$$\left| \frac{1}{m(B)} \int_B f(y)\,dy - f(x) \right| \le \frac{1}{m(B)} \int_B |f(y) - g(y)|\,dy$$

$$\overline{\lim} \qquad + \frac{1}{m(B)} \int_B (g(y) - g(x))\,dy + g(x) - f(x) \Big|$$

(2)

If $x \in E_\alpha$, LHS $> \alpha \Rightarrow$ At least ONE Term on RHS $> \alpha/2$. So $E_\alpha \subset F_{\alpha/2} \cup G_{\alpha/2}$

where $F_\alpha = \{x \mid \underbrace{\overline{\lim} \frac{1}{m(B)} \int_B |f-g|}_{|f-g|^*(x)} > \alpha\}$.

& $G_\alpha = \{x \mid |g-f| > \alpha\}$.

$\Rightarrow m(F_\alpha) \leq \frac{C}{\alpha} \|f-g\|_1 < \frac{C}{\alpha} \cdot \varepsilon$.

Tchebyshev: $m(G_\alpha) < \frac{1}{\alpha} \cdot \|f-g\| < \frac{C}{\alpha} \cdot \varepsilon$.

So $m(E_\alpha) < \frac{C}{\alpha} \cdot \varepsilon$, $\varepsilon \to 0$.

<u>Rmk</u>: Only ever used $\int_B |f|$.

<u>Def</u>: $f \in L^1_{loc}$ "locally integrable" if $\forall$ Ball, $f \cdot \chi_B \in L^1$. E.g. $f(x) = e^{|x|}$.

<u>Non-e.g.</u>: $f(x) = \frac{1}{|x|^d}$. <u>E.g.</u>: $f(x) = \frac{1}{|x|^{d/2}}$.

Thm true with $L^1 \longrightarrow L^1_{loc}$.

Application / Def: If $x\in\mathbb{R}^d$ & $E\in\mathcal{M}$, x is a point of Lebesgue density (for E) if:

$$\lim_{\substack{m(B)\to 0\\ x\in B}} \frac{m(B\cap E)}{m(B)} = 1.$$

E.g. $x\in E^\circ$.

Non.eg.

Cor: $E\in\mathcal{M} \Rightarrow$ (i) a.e. $x\in E$ is of density

(ii) a.e. $x\in E^c$ is not a pt of density.

Pf: $f=\chi_E$. ✓

If $f\in[f]\in L^1_{loc}$, let "Lebesgue set"

$$L_f = \left\{x \,\middle|\, |f(x)|<\infty \ \&\ \lim_{\substack{m(B)\to 0\\ x\in B}} \frac{1}{m(B)}\int_B |f(y)-f(x)|\,dy = 0\right\}$$

Cor: $f\in L^1_{loc}$, $m(L_f^c)=0$.

Pf: Apply LDT to $|f(x)-r|$ for $r\in\mathbb{Q}$.

Let $E_r = \{x \mid \lim_{\substack{x\in B \\ m(B)\to 0}} \frac{1}{m(B)} \int_B |f(y)-r|\,dy = |f(x)-r|\}$

LDT $\Rightarrow m(E_r^c) = 0$.

$\Rightarrow$ let $E = \bigcup_r E_r^c$. has $m(E) = 0$.

Assume $x \notin E$ & $|f(x)| < \infty$. Fix ε, let $|r - f(x)| < \varepsilon$. $\Rightarrow x \in E_r \Rightarrow$

$$\underbrace{\lim \frac{1}{m(B)} \int_B |f(y)-f(x)|\,dy}_{0} \le \overline{\lim} \frac{1}{m(B)} \int_B |f(y)-r|\,dy + |f(x)-r| = 2|f(x)-r| < 2\varepsilon.$$

$\Rightarrow x \in L_f$.

$\Rightarrow L_f^c \subset E \Rightarrow m(L_f^c) = 0$.

Key property needed: Def: $\{U_\alpha\}$ has bounded eccentricity (shrinks regularly) at x if: $U_\alpha \in \mathcal{M}$

(5)

$x \in \bigcap U_\alpha$, $m(U_\alpha) \to 0$, $\exists\, c > 0$:
$\forall U_\alpha\ \exists B$ ball: $x \in B$, $U_\alpha \subset B$ &
$m(U_\alpha) \geq c \cdot m(B)$.

"<u>Lebesgue Differentiation Thm</u>": If $f \in L^1_{loc}$ & U_α shrink reg at x, then $\forall x \in L_f$,

$$\lim \frac{1}{m(U_\alpha)} \int_{U_\alpha} |f(y) - f(x)|\, dy = 0.$$

<u>pf:</u> $x \in L_f$ true with $U_\alpha \rightsquigarrow B$.

But if $U_\alpha \subset B$ & $m(U_\alpha) \geq c\, m(B)$

$$\frac{1}{m(U_\alpha)} \int_{U_\alpha} |f(y) - f(x)|\, dy \leq \underbrace{\frac{1}{c\, m(B)} \int_B |f(y) - f(x)|\, dy}_{\to 0.}$$

$$f(x) \leftarrow \frac{F(x+h) - F(x)}{h} = \frac{1}{m(I)} \int_I f(y)\, dy.$$

$I = (x, x+h)$.