Hwk for 10/25 Due on 10/29.

<u>Recall:</u> $L^1(X,\mu) := \{ f : X \to \mathbb{C} : \int |f| < \infty \} / \sim$

is a metric space (induced from $\|\cdot\|$)

Proved <u>Thm</u> (Reisz-Fischer): <u>is complete.</u>

- Monotone CT: $f_n \geq 0$, $f_n \nearrow f$ a.e. $\Rightarrow \int f_n \to \int f$.
- <u>Fatou:</u> $f_n \geq 0$ $\int \liminf f_n \leq \liminf \int f_n$
  & reverse $f_n \leq g \in L^1 \Rightarrow \limsup \int f_n \leq \int \limsup f_n$.
- <u>Dom Conv Thm:</u> $f_n : X \to \mathbb{C}$, $f_n \to f$ a.e. & $|f_n| \leq g \in L^1 \Rightarrow f_n \xrightarrow{L^1} f$. ($\Rightarrow \int f_n \to \int f$).
- $\{f_n\} \subset L^1$, $f_n \xrightarrow{L^1} f \Rightarrow \exists$ subseq $\{f_{n_k}\}$ $f_{n_k} \to f$ a.e.

$(X_1, \mathcal{M}_1, \mu_1)$, $(X_2, \mathcal{M}_2, \mu_2)$. complete & σ-finite

Want to understand $\mu_1 \times \mu_2$ on $X_1 \times X_2$.

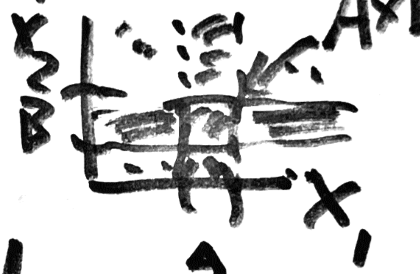
"Obvious" $A \times B \in \mathcal{M}_1 \times \mathcal{M}_2$, "$\mu$", set

$$\mu_0(A \times B) := \mu_1(A) \cdot \mu_2(B). \quad (0 \cdot \infty = 0).$$

(E.g. $X_1 = \mathbb{R} = X_2$, $\mu(\mathbb{R} \times \{0\}) = 0$).

Let $\mathcal{A} = \{$ finite unions of disjoint rectangles $A \times B \in \mathcal{M}_1 \times \mathcal{M}_2 \}$.

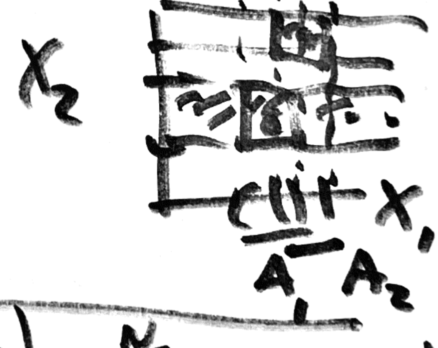
<u>Claim:</u> $\mathcal{A}$ is an algebra.



<u>Pf.</u> $(A \times B)^c = (A^c \times B) \sqcup (A \times B^c) \sqcup (A^c \times B^c) \in \mathcal{A}$.

$((A_1 \times B_1) \cup (A_2 \times B_2))^c$

exercise:

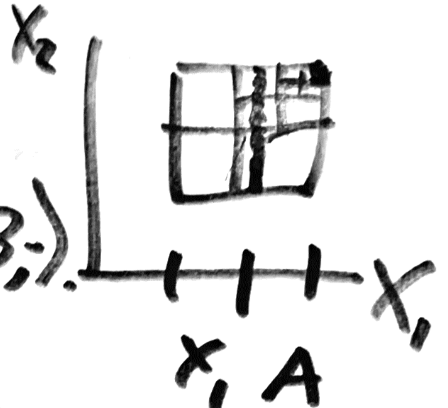


induct

Extend μ_0 to $\mathcal{A}$ by $\mu_0\left(\bigsqcup_{j=1}^{N} A_j \times B_j\right) = \sum_{j=1}^{N} \underbrace{\mu_0(A_j \times B_j)}_{\mu_1(A_j)\mu_2(B_j)}$

<u>Lemma:</u> $(\mathcal{A}, \mu_0)$ is a premeasure.

Pf: If $A\times B = \bigsqcup_{j=1}^{\infty} \underline{A_j\times B_j}$



<u>Want:</u> $\mu_0(A\times B) = \sum_{j=1}^{\infty}\mu_0(A_j\times B_j)$.

Given $x_1\in A$ consider $x_2\in B$ s.t. $(x_1,x_2)\in A\times B$.

Each such x_2 lies in exactly one B_j.

<u>For each x_1</u>, $B = \bigsqcup_{(x_1,x_2)\in A_j\times B_j} B_j \leftarrow \in \mathcal{M}_2$.

Since μ_2 is a meas $\chi_A(x_1)\cdot\mu_2(B) = \sum_{j=1}^{\infty}\chi_{A_j}(x_1)\cdot\mu_2(B_j)$

$$\underbrace{\int_{X_1}\chi_A(x_1)\cdot\mu_2(B)\,d\mu_1(x_1)}_{\mu_0(A\times B)\,=\,\mu_1(A)\cdot\mu_2(B)} \overset{\text{MCT}}{=} \sum_{j=1}^{\infty}\underbrace{\int_{X_1}\chi_{A_j}(x_1)\cdot\mu_2(B_j)\,d\mu_1}_{}$$

$$= \sum_{j=1}^{\infty}\underbrace{\mu_1(A_j)\cdot\mu_2(B_j)}_{\mu_0(A_j\times B_j)}$$

True on $\mathcal{A}$.

$\mu_0 \rightsquigarrow \mu_*$ on $\mathcal{P}(\underbrace{X_1\times X_2}_{X})$, $\mathcal{M} = \overline{\langle\mathcal{A}\rangle}$, $\mu_*|_{\mathcal{M}} = \mu$.

Note $\langle\mathcal{A}\rangle$ is not complete, $N\times Z$

not measble

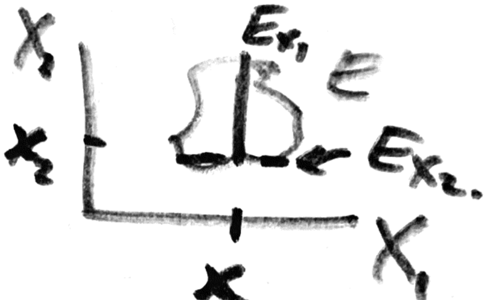
$\leftarrow$ meas 0 in μ_2.

So this $\mu = \mu_1 \times \mu_2$.

Want to compare $\displaystyle\int_{X_1 \times X_2} d(\mu_1 \times \mu_2) \overset{?}{=} \int_{X_1} \int_{X_2} d\mu_2 \, d\mu_1$.

Given $E \in \mathcal{M}$. let $E_{x_1} = \{ x_2 \in X_2 \mid (x_1, x_2) \in E \}$

& $E^{x_2} = \{ x_1 \in X_1 : (x_1, x_2) \in E \}$



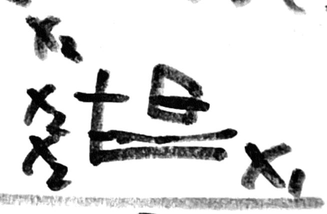
In general, $E^{x_2} \notin \mathcal{M}_1$.

<u>Prop:</u> If $E \in A_{\sigma\delta}$ (= {count intersections in A_σ}, A_σ = {count unions in A}).

(i) Then $\forall x_2$, $E^{x_2} \in \mathcal{M}_1$.

(ii) Moreover, $\mu_1(E^{x_2}) = f(x_2)$ is $\mathcal{M}_2$-measble.

(iii) & $\displaystyle\int_{X_2} \mu_1(E^{x_2}) \, d\mu_2 = \mu(E)$.



<u>pf:</u> I] $E = A \times B$, $E^{x_2} = \begin{cases} A & x_2 \in B \\ \emptyset & \text{oth.} \end{cases} \in \mathcal{M}_1$.

$\mu_1(E^{x_2}) = \begin{cases} \mu_1(A) \\ 0 \end{cases} = \mu_1(A) \cdot \chi_B(x_2)$. $\mathcal{M}_2$-meas ✓.

& $\displaystyle\int_{X_2} \mu_1(A) \chi_B(x_2) \, d\mu_2 = \mu_1(A) \mu_2(B)$.
$= \mu(E)$.

If $E \in \mathcal{A}_\sigma$, $E = \bigcup_{j=1}^\infty \underbrace{A_j \times B_j}_{E_j}$

Can assume $\bigcup_{j=1}^\infty = \bigsqcup_{j=1}^\infty$.

For each x_2, $E^{x_2} = \bigsqcup_{j=1}^\infty \underbrace{E_j^{x_2}}_{\in \mathcal{M}_1} \in \mathcal{M}_1$.

$\Rightarrow \underline{\mu_1(E^{x_2})} = \sum_{j=1}^\infty \mu_1(E_j^{x_2})$.

- each $\mu_1(E_j^{x_2})$ is $\mathcal{M}_2$-meas.
- each $\sum_{j=1}^N \mu_1(E_j^{x_2})$ is.

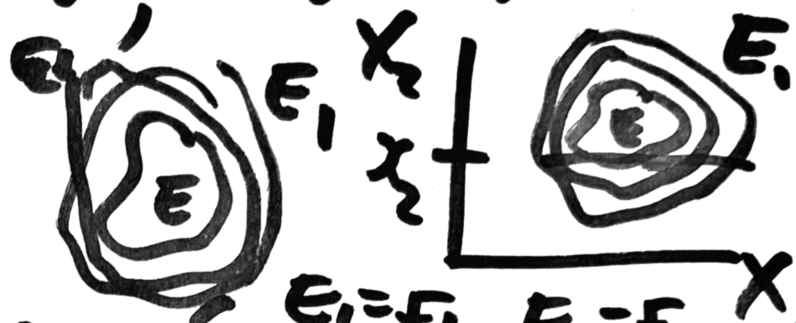
& $\sum^\infty = \lim_{N \to \infty} \sum^N$ so is $\mathcal{M}_2$-meas.

$$\int_{\underline{X_2}} \mu_1(E^{x_2})\, d\mu_2 \overset{MCT}{=} \sum_{j=1}^\infty \underbrace{\int_{X_2} \mu_1(E_j^{x_2})\, d\mu_2}_{\mu(E_j)}.$$

$= \mu(E)$.

Finally, if $E \in \mathcal{A}_{\sigma\delta}$ & $\underline{\mu(E)<\infty}$.
Then $\exists \{E_j\} \subset \mathcal{A}_\sigma$, $E_j \supset E_{j+1} \cdots$ & $\mu(E_1)<\infty$

& $\underline{\bigcap E_j = E}$.



(If F_j not nec nested & $\bigcap F_j = E$, $E_1 = F_1$, $E_2 = F_1 \cap F_2$, ~~$E_1 = \bigcup_{j \ge 1} F_j \in \mathcal{A}_\sigma$~~
~~$E_n = \bigcup_{i \ge n} F_i$~~ & ~~$\bigcap E_n$~~ $\cdots$ (exercise))

Let $f_j(x_2) = \mu_1(E_j^{x_2}) \searrow f(x_2) = \mu_1(E^{x_2})$
as before. & $f_1(x_2) < \infty$ μ_2-a.e. x_2.

f is $\mathcal{M}_2$-measble.

All dominated by $f_1 \Rightarrow$ DCT

$\underline{\mu(E_j) = \int f_j(x_2)\, d\mu_2 \to \int f(x_2)\, d\mu_2 = \mu(E)}$.

To see that $f_1 \in L^1(\mu_2)$, note

that $f_1(x_2) = \mu_1(E_1^{x_2})$ & $E_1 \in \underline{\underline{A_\sigma}}$

$\Rightarrow \int f_1 \, d\mu_2 = \mu(E_1) < \infty$.

If $E \in A_{\sigma\delta}$, general, use

$X_1 = \bigcup F_j$, $\mu_1(F_j) < \infty$, $F_j \subset F_{j+1}$

$X_2 = \bigcup G_j$, $\mu_2(G_j) < \infty$, $G_j \subset G_{j+1}$

$\cancel{\Rightarrow X_1 \times X_2 = \bigcup E_j}$ $\quad F_j \times G_j \in A$.

$\underline{\mu_3(\underbrace{E \cap (F_j \times G_j)}_{\in A_{\sigma\delta}})} < \infty$.

Can apply prev results + MCT.

Prop: If $E \in \mathcal{M}$ then

(i) E^{x_2} is μ_1-meas for μ_2-a.e. x_2.

(ii) $\mu_1(E^{x_2})$ is $\mathcal{M}_2$-meas - - - -

(iii) $\int_{X_2} \mu_1(E^{x_2}) \, d\mu_2 = \mu(E)$. (7)

Pf: $E \in \mathcal{M}$, Recall $\exists F \in \mathcal{A}_{\sigma\delta}$ s.t.
$\mu(F \setminus E) = 0$. i.e. $F = E \sqcup Z$
Know for F. For Z, $\exists G \in \mathcal{A}_{\sigma\delta}$
s.t. $Z \subset G$ & $\mu(G) = 0$. & $Z^{x_2} \subset G^{x_2}$.
But $\mu_1(G^{x_2}) = 0 \Rightarrow \overset{\cap}{\mathcal{M}_1}$.

$$\underline{\mu_1(E^{x_2})} = \underline{\mu_1(F^{x_2})} - \underbrace{\mu_1(Z^{x_2})}_{=0} \quad ✓.$$

Thm (Fubini–Tonelli): Let $f \in L^1(X_1 \times X_2)$.

Then: (i) μ_2-a.e. x_2, slice

$$f^{x_2}(x_1) = f(x_1, x_2) \in L^1(X_1, \mu_1).$$

(ii) $$\int_{X_1} f(x_1, x_2)\, d\mu_1(x_1) \in L^1(X_2, \mu_2).$$

(iii) $$\int_{X_2} \left(\int_{X_1} f(x_1, x_2)\, d\mu_1 \right) d\mu_2 = \int_{X_1 \times X_2} f \, d(\mu_1 \times \mu_2).$$