Recall: $(X, \mathcal{A}, \mu_0) \rightsquigarrow \mu_* \rightsquigarrow \underset{\uparrow}{\mathcal{M}}$, $\mu = \mu_*|_{\mathcal{M}}$.

human definition

Uniqueness

Could there be another $(X, \mathcal{B}, \nu)$ with $\nu = \mu_*|_{\mathcal{B}}$. Claim: $\mathcal{B} \subset \mathcal{M}$.

Fact 1: $\mu_*(E) = \inf\limits_{E \subset F \in \mathcal{A}_\delta} \mu_*(F)$. $\quad \mathcal{A}_\delta = \{\bigcap_{j=1}^{\infty} A_j \mid A_j \in \mathcal{A}\}$,

Fact 2: $E \in \mathcal{M} \iff \underbrace{\forall \varepsilon > 0 \ \exists F \overset{\supset E}{\in} \mathcal{A}_\delta \text{ with } \mu_*(F \setminus E) < \varepsilon}$,

($\iff \exists F \in \mathcal{A}_\delta$ with $\mu_*(F \setminus E) = 0$).

Thm: $\mathcal{B} \subset \mathcal{M}$. pf: If not, $\exists E \in \mathcal{B} \setminus \mathcal{M}$.

Then $\exists \varepsilon > 0 : \forall F \supset E,\ F \in \mathcal{A}_\delta,\ \mu_*(F \setminus E) \geq \varepsilon$.

But $\nu(E) = \mu_*(E) \Rightarrow \exists F \overset{\in \mathcal{A}_\delta}{\supset E}$ with $\underset{\nu(F)}{\underset{\parallel}{\mu_*(F)}} \leq \underset{\nu(E)}{\underset{\parallel}{\mu_*(E)}} + \frac{\varepsilon}{2}$.

But $\nu(F) = \underbrace{\nu(E)}_{\geq \nu(F) - \frac{\varepsilon}{2}} + \underbrace{\nu(F \setminus E)}_{\geq \varepsilon} \geq \nu(F) + \frac{\varepsilon}{2}$. $\times$

Lebesgue integration: $(X, \mathcal{M}, \mu)$ measure space.

$f : X \to \overline{\mathbb{R}}_{\geq 0} = [0, \infty]$. f measurable, i.e. $f^{-1}(\text{open}) \in \mathcal{M}$.

Def: f is simple if $f(X)$ is finite, $f(X) = \{c_1, \dots, c_N\} \subset \mathbb{R}$.

$E_j = f^{-1}(\{c_j\}) \in \mathcal{M}$, $f = \sum_{j=1}^{N} c_j \chi_{E_j}$. $\leftarrow$ not unique.

Def: Canonical form for f: $f(X) = \{c_1, \ldots, c_N\}$, $c_i \neq c_j \neq 0$.

$$f = \sum_{j=1}^{N} c_j \chi_{E_j}: \quad E_j \text{ pairwise disjoint} \qquad 0 = \chi_E - \chi_E.$$

Def: Given simple function $\varphi^{\geq 0}$ in canonical form,

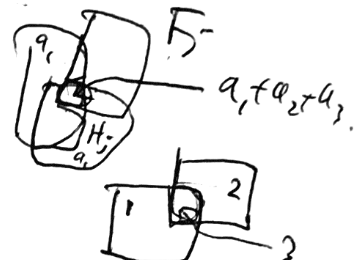
$$\int \varphi \, d\mu = \sum_{j=1}^{N} c_j \, \mu(E_j).$$

Lemma: Def is indep of representation, i.e. if $\varphi = \sum_{j=1}^{M} a_j \chi_{F_j}$,

$$\int \varphi \, d\mu = \sum_{j=1}^{M} a_j \, \mu(F_j)$$

Convention $0 \cdot \infty = 0$.

pf: Breaking up F_j to disjoint H_k's. F_j



For each value c_j, let $E_j = \bigsqcup_{c_j = \sum a_{H_k}} H_k$.

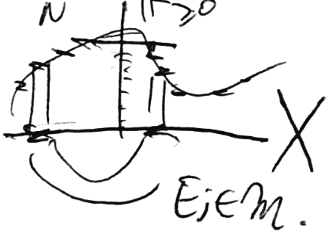
So $$\sum_{j=1}^{M} a_j \, \mu(F_j) = \sum_{j=1}^{N} c_j \cdot \sum_{c_j = \sum a_{H_k}} \mu(H_k) = \sum_{j=1}^{N} c_j \, \mu(E_k).$$

Cor: ① $\int(\varphi_1 + \varphi_2) = \int \varphi_1 + \int \varphi_2$. ② $\int c\varphi = c \int \varphi$.

③ If $E \cap F = \emptyset$, $\int_{E \cup F} \varphi = \int_E \varphi + \int_F \varphi$. $= \int \varphi \chi_{E \cup F}$. ← still simple.

④ If $\varphi_1 \leq \varphi_2 \Rightarrow \int \varphi_1 \leq \int \varphi_2$. ⑤ If φ complex valued, $\left|\int \varphi\right| \leq \int |\varphi|$.

Recall $f: X \to [0, \infty]$ measurable $\Rightarrow \exists \, \varphi_k \leq \varphi_{k+1} \to f$.



Def 0: $\int f \, d\mu := \lim \int \varphi_k \, d\mu$.

Def: $\int f\,d\mu = \sup_{0\le\varphi\le f} \int \varphi\,d\mu$, if $<\infty$, we say f is Lebesgue-integrable, $f\in L^1(X,d\mu)$.

$\int \varphi(x)\,\mu(dx)$.

Monotone Convergence Thm: If $f_n \geq 0$, $f_n \le f_{n+1}$ & $f_n \nearrow f$. p.w.

then $\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$.

pf: $f_n(x)$ increasing, so have a limit (possibly ∞).

Clearly $\int f_n \le \int f$. $\Rightarrow \lim \int f_n \le \int f$, need other direction.

Fix $\alpha\in(0,1)$. let $0\le\varphi\le f$ be any simple function,

& set $E_n := \{x\in X \mid f_n(x) \ge \alpha\cdot\varphi(x)\} \nearrow X$.

each $E_n \in \mathcal{M}$.

$$\int f_n \ge \int f_n \chi_{E_n} \ge \int \alpha\varphi\chi_{E_n},$$

Note: if $\varphi = \sum a_i \chi_{F_i}$ then $\varphi\chi_{E_n} = \sum_{i=1}^{N} a_i \chi_{F_i\cap E_n}$.

So $\int \varphi\chi_{E_n} = \sum a_i\, \mu(F_i\cap E_n) \to \mu(F_i)$. Limit $n\to\infty$:

$\lim \int f_n \ge \alpha \cdot \lim \int \varphi\chi_{E_n} = \alpha \int\varphi$. True for all $\varphi\le f$, so true for sup

$\lim_{n\to\infty} \int f_n \ge \alpha \sup_{\varphi\le f} \int\varphi = \alpha\cdot\int f$. True for all $\alpha<1$, $\alpha\to 1$.

Rmks: Dropping monotonicity fails E.g.: $f_n = \chi_{[n,n+1]}$.

E.g.: $f_n = \chi_{[0,\frac{1}{n}]}\cdot n$.

Cor: If $\{f_n\} \geq 0$, then $\int \sum_n f_n = \sum_n \int f_n$

pf: Start with f_1 & f_2. $\underset{\text{simple.}}{\exists} \{\varphi_j\} \nearrow f_1$, $\{\psi_j\} \nearrow f_2$.

$\varphi_j + \psi_j$ simple, increasing to $f_1 + f_2$. Monotone Conv $\Rightarrow$

$$\int (f_1 + f_2) = \lim_j \int (\varphi_j + \psi_j) = \lim_j \left(\int \varphi_j + \int \psi_j \right) = \int f_1 + \int f_2 .$$

$\Rightarrow \int \underbrace{\sum_1^N f_n}_{F_N} = \sum_1^N \int f_n$. By MCT

$F_N \nearrow F = \sum_1^\infty f_n$. $\underset{\int \sum_n f_n}{\int F} = \lim_N \int F_N = \lim_N \sum_1^N \int f_n$

$= \sum_1^\infty \int f_n$.

Prop: $f \geq 0 \Rightarrow$ $\int f = 0 \Leftrightarrow f = 0$ a.e.

pf: If f simple, then $\sum a_j \mu(E_j) = 0 \Leftrightarrow \mu(E_j) > 0 \Rightarrow a_j = 0$.

For general f, pf($\Leftarrow$): $f = 0$ a.e., let $0 \leq \varphi \leq f \Rightarrow \varphi = 0$ a.e.

$\Rightarrow \int \varphi = 0 \Rightarrow \sup_\varphi \int \varphi = 0 = \int f$.

pf $\Rightarrow$: $\{x \mid f(x) > 0\} = \bigcup_{n=1}^\infty E_n$, $E_n := \{x \mid f(x) \geq \frac{1}{n}\}$.

If $\underbrace{f \neq 0 \text{ a.e.}}_{\neg (f = 0 \text{ a.e.})}$, then $\exists n$ s.t. $\mu(E_n) > 0$. Then

$f \geq \frac{1}{n} \cdot \chi_{E_n}$. $\int f \geq \int \frac{1}{n} \chi_{E_n} = \frac{1}{n} \mu(E_n) > 0$.

$f \geq f \cdot \chi_{E_n} \geq \frac{1}{n} \cdot \chi_{E_n}$.

(4)

Thm (MCT): If $f_n \geq 0$, $\underbrace{f_n \nearrow f}_{f_n(x) \leq f_{n+1}(x)}$ a.e. Then

$\int f_n \to \int f$.

pf: Let $E = \{x \mid f_n(x) \to f(x)\}$. so $\mu(E^c) = 0$.

so $f - f\chi_E = f\chi_{E^c} = 0$ a.e. & $f_n - f_n\chi_E = f_n \cdot \chi_{E^c} = 0$ a.e.

$\Rightarrow \int(f - f\chi_E) = 0 \Rightarrow \int f = \int f\chi_E$. & $\int f_n = \int f_n\chi_E$.

But $f_n\chi_E \nearrow f\chi_E$ p.w. $\Rightarrow \underbrace{\int f_n = \int f_n\chi_E} \to \int f\chi_E = \int f$.

Fatou's Lemma: $\{f_n \geq 0\}$ Not nec. monotonic $\Rightarrow$

$$\int(\liminf f_n) \leq \liminf \int f_n.$$

pf: $\forall k \geq 1$, $\forall j \geq k$ $\forall x$, $\inf_{n \geq k} f_n(x) \leq f_j(x)$. $\Rightarrow$

$$\underbrace{\int \inf_{n \geq k} f_n(x)}_{= F_k} \leq \int f_j \quad (\text{true } \forall j \geq k).$$

$F_k(x) \nearrow F(x) = \liminf f_m(x)$ $\quad \forall j \geq k$.

$$\Rightarrow \int \liminf f_n = \int F = \lim_{k\to\infty} \int F_k \leq \liminf_{k\to\infty} \int f_k$$

$$\leq \lim_{k\to\infty} \inf_{j \geq k} \underbrace{\int f_j}_{\forall j \geq k}.$$