Recall: Thm: (X,d) metric $E \subset X$. TFAE:

1. E complete, & totally bdd.
2. Bolzano-Weierstrass every seq has convergent subseq.
3. E is compact.

---

(3) ⇒ (2) pf ¬(2) ⇒ ¬(3): Say $\exists \{x_n\} \subset E$ with no convergent subseq. Every $x \in E$ is not a limit pt of $\{x_n\}$. I.e. $\forall x \in E\ \exists \delta_x > 0$, $B_{\delta_x}(x)$ intersecting only finitely many x_n's.

Then $E \subset \bigcup_{x \in E} B_{\delta_x}(x)$ ~~has~~ can't have finite subcover.

Cor: E cpt ⇒ cardinality $\leq \mathfrak{c}$ continuum.

pf: Claim: E cpt ⇒ separable (countable dense subset).

pf: $E \subset \bigcup_{j=1}^{N_n} B_{1/n}(x_j^{(n)})$, $\forall n \geq 1$, ⇒ $\{x_j^{(n)}\}$ is countable & dense

$\Rightarrow E = \overline{\{x_j^{(n)}\}}$.

Thm: $(X, \mathcal{A}, \mu_0)$ premeasure ⇒ $\mu_*(E) = \inf \sum \mu_0(A_j)$, $E \subset \bigcup_{j=1}^{\infty} A_j$, $A_j \in \mathcal{A}$

algebra ← complements & finite unions

Def: $E \subset (X, \mu_*)$ is μ_*-measurable if $\forall A \subset X$, $\mu_*(A) = \mu_*(A \cap E) + \mu_*(A \cap E^c)$

Lemma: If μ_* came from μ_0, then every $A \in \mathcal{A}$ is μ_*-measurable.

If $E \in \mathcal{A}$ then $\mu_*(E) = \mu_0(E)$.

Thm: ① $\mathcal{M} = \{\mu_*\text{-measurable sets}\}$ is a σ-alg &

② $\mu = \mu_*\big|_{\mathcal{M}}$ is a measure. $\big(\Rightarrow (E \in \mathcal{A}) \Rightarrow \mu(E) = \mu_0(E)\big)$

③ μ is complete. (Every $A \subset E \in \mathcal{M}$ with $\mu(E) = 0$ $\Rightarrow A \in \mathcal{M}$).

---

pf: $E \in \mathcal{M} \iff E^c \in \mathcal{M}$, $\quad E_1, E_2 \in \mathcal{M} \Rightarrow E_1 \cup E_2 \in \mathcal{M}$.

Claim: μ is finitely additive: If $E_1 \cap E_2 = \emptyset$, $E_1, E_2 \in \mathcal{M}$.

$$\mu(E_1 \cup E_2) \overset{E_1 \in \mathcal{M}}{=} \mu(\underbrace{(E_1 \cup E_2) \cap E_1}_{E_1}) + \mu(\underbrace{(E_1 \cup E_2) \cap E_1^c}_{E_2}).$$

First assume $E_1, \dots, E_n, \dots \in \mathcal{M}$, pairwise disjoint, let $A \subset X$ be arbitrary. Let $B_n = \bigcup_{j=1}^{n} E_j \in \mathcal{M}$, let $B = \bigcup_{j=1}^{\infty} E_j$.

Want: $$\mu_*(A) = \mu_*\Big(A \cap \big(\bigcup E_j\big)\Big) + \mu_*\Big(A \cap \big(\bigcup E_j\big)^c\Big).$$

Know: $$\mu_*(A) = \mu_*(A \cap B_n) + \mu_*(A \cap B_n^c).$$

$$\overset{E_n \in \mathcal{M}}{=} \mu_*(\underbrace{(A \cap B_n) \cap E_n}_{A \cap E_n}) + \mu_*(\underbrace{(A \cap B_n) \cap (E_n^c)}_{A \cap B_{n-1}}) + \mu_*(A \cap B_n^c)$$

$$= \sum_{j=1}^{n} \mu_*(A \cap E_j) + \mu_*(A \cap B_n^c).$$

$$\geq \sum_{j=1}^{n} \mu_*(A \cap E_j) + \mu_*(A \cap B^c), \quad n \to \infty.$$

$$\boxed{\mu_*(A) \geq \sum_{j=1}^{\infty} \mu_*(A \cap E_j) + \mu_*(A \cap B^c)}$$

$$\geq \mu_*(A \cap B) + \mu_*(A \cap B^c) \geq \mu_*(A).$$

Applies with $A = B \in \mathcal{M}$, $\underbrace{\mu(B) = \sum_{j=1}^{\infty} \mu(E_j)} + \cancel{\mu(B \cap B^c)}$

Claim: Let $\cancel{Z \subset E \in \mathcal{M}}$ with $\underline{\mu_*(Z) = 0}$. Let $A \subset X$ be arbitrary.

$$\mu_*(A) \le \mu_*(A \cap Z) + \mu_*(A \cap Z^c)$$
$$\le \cancel{\mu_*(Z)}^{0} + \mu_*(A)$$
$$\Rightarrow Z \in \mathcal{M}.$$

$$(X, \mathcal{A}, \mu_0) \rightsquigarrow \mu_* \rightsquigarrow (X, \mathcal{M}, \mu).$$ $\mathcal{O}$ open sets.

$$\mathcal{M} \supset \underline{\sigma\text{-alg}\langle \mathcal{A} \rangle} = \mathcal{B}$$ ← Borel σ-alg.

Def: $(X, \mathcal{A}, \mu_0)$ is σ-finite if $\exists$ countable $E_j \in \mathcal{A}$, $X = \bigcup E_j$, $\mu_0(E_j) < \infty$.

Thm: (Uniqueness of Caratheodory extension). Let $(X, \mathcal{A}, \mu_0)$ be premeasure, $\underline{\sigma\text{-finite}}$, & $(X, \mathcal{M}, \mu)$ the extension. The extension is unique, i.e. if $(X, \mathcal{B}, \nu)$ is another measure with $\mathcal{M} \supseteq \mathcal{B} \supset \mathcal{A}$ & $\forall A \in \mathcal{A}$, $\nu(A) = \mu_0(A) \Rightarrow$ $\forall E \in \mathcal{B}$ $\nu(E) = \mu(E)$, i.e. $\mu|_{\mathcal{B}} = \nu$.

---

pf: Let $F \in \mathcal{B}$ with $\underline{\mu(F) < \infty}$.

Claim: $\nu(F) = \mu(F)$. Let $F \subset \bigcup E_j = E$, $E_j \in \mathcal{A}$ with $\mu(E) \le \mu(F) + \varepsilon$. Then $\underline{\nu(F)} \le \sum_{j=1}^{\infty} \nu(E_j) = \underbrace{\sum_{j=1}^{\infty} \mu_0(E_j)}$

$$\Rightarrow \nu(F) \le \mu(F).$$

$$\nu(E) = \lim_{n\to\infty} \nu\left(\bigcup_{i=1}^{n} E_i\right) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} E_i\right) = \mu(E).$$

$$\mu(F) \le \mu(E) = \nu(E) = \nu(F) + \nu(E\setminus F)$$

$$\le \nu(F) + \underbrace{\mu(E\setminus F)}_{<\varepsilon} \quad \Rightarrow \mu(F) = \nu(F).$$

For F arbitrary, $F = \bigcup (\underline{F \cap E_i})$, $E_i \in \mathcal{A}$ cover X, $\mu(E_i) < \infty$.