Recall: Char functions: $f = \chi_E \leftarrow \in \mathcal{M}$.

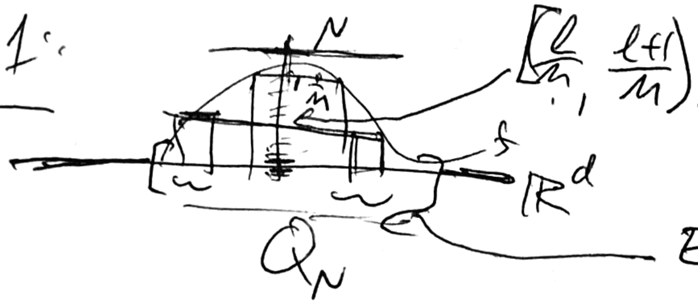
Simple functions $f = \sum_{j=1}^{N} a_j \chi_{E_j}$

Step functions: $f = \sum_{j=1}^{N} a_j \chi_{R_j} \leftarrow$ rectangles.

Proved 1: Given f meas'ble, $\exists\, \varphi_k$ simple functions, $|\varphi_k(x)| \le |\varphi_{k+1}(x)|$, $\varphi_k \to f$ pointwise.

Proved 2: Given $f \in \mathcal{M}$, $\exists\, \psi_k$ step functions, $\psi_k \to f$ a.e.

Recall pf of 1:



$0 \le \ell \le NM$

$F_{N,M}$

$\varphi_k = F_{2^k,2^k}$.

Littlewood's Principles. (1) Every meas'ble set is "nearly" finite union of intervals. (in $\mathbb{R}$).

(2) Every measurable function is nearly continuous.

(3) Every convergent sequence is nearly uniformly so.

Thm (Egorov): $\{f_k\}$ seq of meas'ble functions on E, with $m(E) < \infty$. Assume a.e. $x \in E$, $f_k(x) \to f(x)$. Then $\forall \varepsilon > 0$ $\exists\, A_\varepsilon$ closed $\subset E$ with $m(E \setminus A_\varepsilon) < \varepsilon$ & with $f_k \to f$ uniformly on A_ε.

Pf: Enough to assume $f_k(x) \to f(x)$ $\forall x \in E$. For each n, k,

$$E_{n,k} = \left\{ x \in E \;\middle|\; |f_j(x) - f(x)| < \frac{1}{n},\ \forall j > k \right\}$$

For fixed n, $E_{n,k} \subset E_{n,k+1} \subset \cdots$ $E_{n,k} \nearrow E$.

Since $m(E) < \infty$, $\exists k_n$ s.t. $m(E \setminus E_{n,k_n}) < \frac{1}{2^n}$

Let $\tilde{A}_\varepsilon = \bigcap_{n \geq N} E_{n,k_n}$, where N is large enough that $\underbrace{\sum_{n \geq N} \frac{1}{2^n} < \varepsilon}$. __Claim:__ on $\tilde{A}_\varepsilon$, $f_k \to f$ uniformly.

i.e. Want: $\forall \varepsilon' > 0$, $\exists N'$ s.t. $\forall k > N'$, $|f_k(x) - f(x)| < \varepsilon$ $\forall x \in \tilde{A}_\varepsilon$

Note: $m(E \setminus \tilde{A}_\varepsilon) \leq \sum_{n \geq N} m(E \setminus E_{n,k_n}) < \varepsilon$

Pf (Claim): Given $\varepsilon' > 0$, let $N' \geq \max\left(\frac{1}{\varepsilon'}, k_N\right)$. $\forall k > N'$, $\forall x \in \tilde{A}_\varepsilon$

$\Rightarrow x \in E_{n,k_n}$ for some $n > N'$ & $k_n > N'$. $\underset{j > k_n > N'}{\Rightarrow} |f_j(x) - f(x)| < \frac{1}{n} < \frac{1}{N'} < \varepsilon$

$\tilde{A}_\varepsilon$ is not necessarily closed, but $\exists A_\varepsilon$ closed $\subset \tilde{A}_\varepsilon$ with $m(\tilde{A}_\varepsilon \setminus A_\varepsilon) < \varepsilon$. $\Rightarrow m(E \setminus A_\varepsilon) < \varepsilon$.

To ②: **Thm (Lusin):** Let f be measurable on E. ($\& |f| < \infty$), assume $m(E) < \infty$. Then $\forall \varepsilon > 0$ $\exists$ F_ε closed $\subset E$ s.t. $m(E \setminus F_\varepsilon) < \varepsilon$ s.t. $f\Big|_{F_\varepsilon}$ is cont.

Note: $\not\Rightarrow$ $f|_E$ is cont at all pts of F_ε

F_ε

E

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pf: Thm $\Rightarrow$ $\exists$ f_n seq of step functions $f_n \to f$ a.e. $x \in E$, $f_n = \sum_{j=1}^{M_n} a_j \chi_{R_j}$

R_j

For each f_n, $\exists$ E_n with $m(E_n) < \frac{1}{2^n}$ s.t. $f_n\Big|_{E_n^c}$ is continuous. By Egorov $\exists$ A_ε closed $\subset E$ with $m(E \setminus A_\varepsilon) < \varepsilon$ & $f_n \to f$ uniformly on A_ε.

Consider $F' = A_\varepsilon \setminus \bigcup_{n>N} E_n$ where N satisfies $\sum_{n>N} \frac{1}{2^n} < \varepsilon$. Then $m(A_\varepsilon \setminus F') < \varepsilon$. & $\forall n>N$, f_n are continuous & converge uniformly $\Rightarrow$ f is cont on F'.

Now let $F \subset F'$ closed & $m(F' \setminus F) < \varepsilon$.

More general setup: X = abstract set

An algebra $\mathcal{A}$ on X is a set of subsets of X s.t. $\mathcal{A}$ is closed under complements & finite unions & finite intersections, & $\emptyset \in \mathcal{A}$

Def: A premeasure μ_0 on $(X, \mathcal{A})$, $\mu_0 : \mathcal{A} \to [0,\infty]$ s.t. $\mu_0(\emptyset) = 0$, & μ_0 is countably additive if $E_1, \dots, E_n, \dots \in \mathcal{A}$ & pairwise disjoint & if $\bigcup E_j = E \in \mathcal{A}$

$$\Rightarrow \mu_0(E) = \sum_j \mu_0(E_j).$$

Def: An exterior (outer) measure μ_* on X is $\mu_* : \mathcal{P}(X) \to [0,\infty]$, s.t. $\mu_*(\emptyset) = 0$, monotone, $\mu_*(E_1) \le \mu_*(E_2)$ whenever $E_1 \subset E_2$. & countably subadditive, $\mu_*\left(\bigcup E_j\right) \le \sum_j \mu_*(E_j)$.