Countable: S is countable iff $\exists\, f: S \to \mathbb{N}$ bijection, i.e. $f(s_1)=f(s_2) \Rightarrow s_1 = s_2$ & $\forall n \in \mathbb{N}$ $\exists\, s \in S: f(s)=n$.

$\mathbb{Z}$ countable: $0, 1, -1, 2, -2, 3, -3, \ldots$

$\mathbb{Q}$ countable $h\left(\frac{p}{q}\right) = |p| + q$, $q > 0$

$h=1 \Rightarrow p=0,\ q=1$

$h=2 \Rightarrow p=\pm 1,\ q=1$

$$\begin{array}{cccc} \frac{0}{1} & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} \\ \frac{0}{2} & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} \\ & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} \end{array}$$

$\mathbb{R}$ uncountable $\supset S = \{0.a_1 a_2 a_3 \ldots \mid a_i \in \{0,1\}\}$.

$f^{-1}(1) = 0.\boxed{0}011010\ldots$

$f^{-1}(2) = 0.1\boxed{1}011010\ldots$

$f^{-1}(3) = 0.00\boxed{0}0001\ldots$

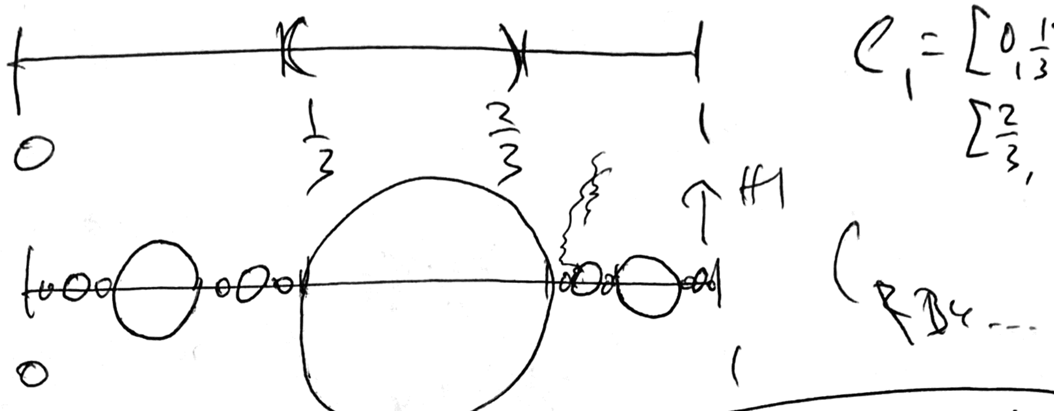
Claim: $\exists\, s \in S$ s.t. $\nexists\, n \in \mathbb{N}$ with $f^{-1}(n) = s$.

pf: $s = 0.101\ldots \in S$.

∂E = boundary of E = $\overline{E} \setminus E^{\circ}$.

Def: E is **perfect** if closed & has no isolated pts.

Def: $C \subset [0,1]$ is the Cantor set:



$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.

C_2 ...

$$\bigcap_j C_j = C.$$

(perfect)

Tot disconnected: $\forall\, x \lneq y \in C\ \exists z \notin C$: $x < z < y$.

Rectangle $R = [a_1, b_1] \times \cdots \times [a_d, b_d] \subseteq \mathbb{R}^d$.

$\mathrm{vol}(R) = |R| = (b_1 - a_1) \cdots (b_d - a_d)$.

Def: $R = \bigsqcup^{\text{almost}}_j R_j$ **if** $R_j^{\circ} \cap R_i^{\circ} = \emptyset$.

Lemma: If $R = \bigsqcup_{j=1}^{N} R_j \Rightarrow \mathrm{vol}(R) = \sum_{j=1}^{N} |R_j|$

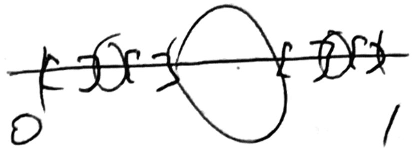
Pf.

Lemma: If $R \subset \bigcup_{j=1}^{N} R_j \Rightarrow |R| \leq \sum_{j=1}^{N} |R_j|$.

Lemma: ~~$\exists$~~ $m(\mathcal{C}) = 0$. $\mathcal{C} = \bigcap \mathcal{C}_j$.

$\mathcal{C}_j$: finite union of rect.

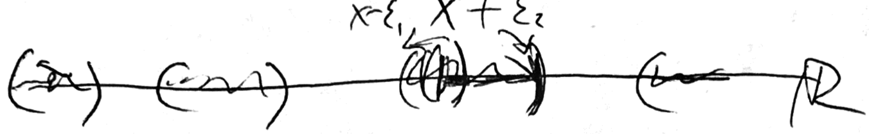
pf: $|\mathcal{C}_j| = \dfrac{2^j}{3^j} \to 0$.



$\#$ intervals in $\mathcal{C}_j$ $= 2^j$.

$|R_j| = \dfrac{1}{3^j}$.

In $\mathbb{R}^1$: **Thm:** Every $\mathcal{O} \subset \mathbb{R}$ open is a countable union of open intervals.

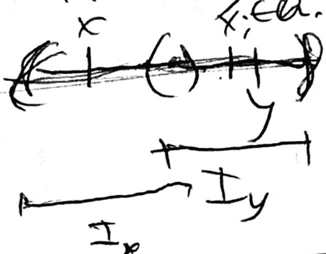


pf: If $x \in \mathcal{O}$, $\exists$ $(x-\varepsilon_1, x+\varepsilon_2) \subset \mathcal{O}$.

$\exists$ $\varepsilon_1(x)$, $\varepsilon_2(x) =$ ~~the~~ $\sup\{\varepsilon > 0 \mid (x, x+\varepsilon) \subset \mathcal{O}\}$.

$\Rightarrow \mathcal{O} = \bigcup_{x \in \mathcal{O}} I_x$, where $I_x = (x - \varepsilon_1(x), x + \varepsilon_2(x))$.

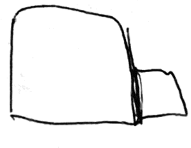
If $I_x \cap I_y \neq \emptyset$ claim: $I_x = I_y$.



By sup of $\varepsilon_1(x), \ldots, \varepsilon_2(y)$.

Each interval contains a rat'l.

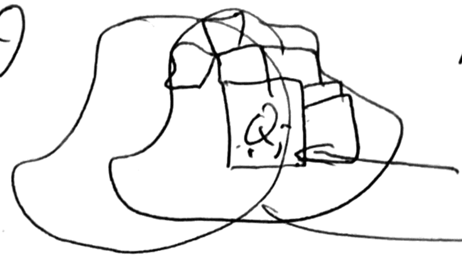
$\mathcal{O} = \bigsqcup_{i=1}^{\infty} I_{x_i}$.



Note: For $\mathbb{R}$, this is sufficient

$$m(\mathcal{O}) = m\Big(\bigsqcup I_j\Big) = \sum |I_j| .$$

What about $\mathbb{R}^d$? $\mathcal{O}$



$m(\mathcal{O}) \overset{?}{=} \sum |Q_j|$.

Notion of Exterior measure m_* on all subsets of $\mathbb{R}^d$.

$$m_*(\underset{\subset \mathbb{R}^d}{E}) := \inf_{E \subset \bigcup_{j=1}^{\infty} Q_j} \sum_{j=1}^{\infty} |Q_j|$$

$E \in \bigsqcup_{j=1}^{\infty} Q_j$

need ∞.

Lemma: $m_*(\{x\}) = 0$ $\quad Q = [x]$.

Lemma: $m_*(\underset{\text{closed cube}}{Q}) = |Q|$. pf: $m_*(Q) \le |Q|$.

Q

let $Q \subset \bigcup Q_j$. Fix $\varepsilon > 0$. ~~For~~ each j, $\exists\, S_j \overset{\text{open}}{\supset} Q_j$ with $|S_j| \le |Q_j|(1+\varepsilon)$ $\cancel{+\frac{\varepsilon}{2^j}}$. $Q \subset \bigcup Q_j \subset \bigcup S_j$

& Q cpt $\Rightarrow Q \subset \bigcup_{j=1}^{N} S_j$. $\quad |Q| \le \cancel{\Big|\bigcup_{j=1}^{N} S_j\Big|} \overset{\text{rectangle lemma}}{\le} \sum_{j=1}^{N} |S_j|$

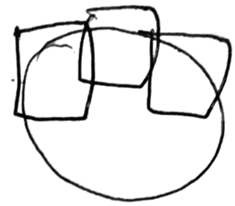
$\le \sum_{i=1}^{\infty} |Q_i| (1+\varepsilon)$.

$\Rightarrow |Q| \le (1+\varepsilon)\, m_*(Q)$.

Q

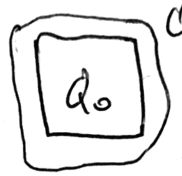
Lemma: $\mathcal{O}$ = open cube $= Q^{\circ}$. $\Rightarrow m_*(\mathcal{O}) = |Q|$.

pf ($\leq$) $\mathcal{O} \subset Q \Rightarrow m_*(\mathcal{O}) \leq |Q|$.



pf ($\geq$): Let $\mathcal{O} \subset \bigsqcup_{j=1}^{\infty} Q_j$. $\exists\, Q_0 \subset \mathcal{O}$ closed cube

with $|Q_0| \geq |\mathcal{O}| - \varepsilon$.



$$Q_0 \subset \bigsqcup Q_j \Rightarrow \inf_{Q_0 \subset \sqcup Q_j} \sum |Q_j| \geq |Q_0| \geq |\mathcal{O}| - \varepsilon$$

Note: If $E \subset \mathbb{R}^d \Rightarrow \forall \varepsilon > 0\ \exists\, E \subset \bigcup_{j=1}^{\infty} Q_j$ s.t.

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \varepsilon$$

Cor: (monotonicity): $E_1 \subset E_2 \Rightarrow m_*(E_1) \leq m_*(E_2)$.

pf: Any cover of E_2 covers E_1. Use Note.

Cor: If E bdd $\Rightarrow m_*(E) < \infty$. pf: $E \subset Q$.

Lemma: (Countable sub-additivity): If $E = \bigcup_{j=1}^{\infty} E_j$, $\boxed{m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)}$

pf: Assume $m_*(E) < \infty$. Fix $\varepsilon > 0$. Let $E_j \subset \bigcup_{k=1}^{\infty} Q_{j,k}$. Can find such with $\sum_j \sum_k |Q_{j,k}| \leq \sum_j m_*(E_j) + \sum_j \frac{\varepsilon}{2^j}$. $m_*(E) \leq \sum_j \sum_k |Q_{j,k}|$

But then $\bigcup_{j,k} Q_{j,k} \supset \bigcup E_j \supset E$. $\Rightarrow$ $\cancel{m_*(E)}$ $\leq \sum m_*(E_j) + \varepsilon$

Lemma: Let $E \subset \mathbb{R}^d$ be ~~arb~~ arbitrary. Then $m_*(E) = \inf\limits_{E \subset O \text{ open}} m_*(O)$.

pf: Then $m_*(E) = \inf\limits_{E \subset O_\varepsilon} \sum m_*(O_\varepsilon)$.

Lemma: $E \subset \mathbb{R}^d$ arbitrary. Then $m_*(E) = \inf\limits_{E \subset O \text{ open}} m_*(O)$.

pf. $E \subset O \Rightarrow m_*(E) \le m_*(O)$. Need $\ge$.

~~Say~~ want $E \subset O$. Take $E \subset \bigcup Q_j$, $\sum_j m_*(Q_j) \le m_*(E) + \frac{\varepsilon}{2}$.

For each j, $Q_j \subset O_j$ with $|O_j| \le |Q_j| + \frac{\varepsilon}{2^{j+1}}$.

Then $O = \bigcup O_j \supset \bigcup Q_j \supset E$.

$$m_*(O) \le \sum_j m_*(O_j) \le \sum_j m_*(Q_j) + \frac{\varepsilon}{2} \le m_*(E) + \varepsilon.$$