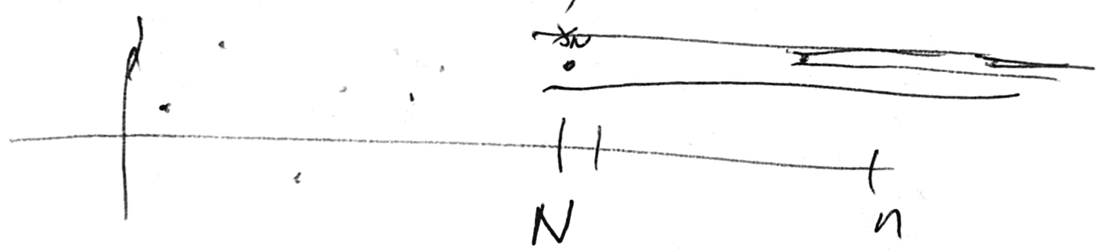
$\mathbb{N}$: $0=\{\}$, $1=\{\{\}\}$, $2=\{\{\},\{\{\}\}\}$

$3=\{\underbrace{\{\}}, \underbrace{\{\{\}\}}, \underbrace{\{\{\},\{\{\}\}\}}\}$ ...

$\mathbb{Z}=\mathbb{N}\cup -\mathbb{N}$, $\mathbb{Q}=\left\{\frac{p}{q} \;\middle|\; p,q\in\mathbb{Z},\ q>0,\ (p,q)=1\right\}$.

<u>Def</u>: $\{x_n\}\subset\mathbb{Q}$ is <u>Cauchy</u> if: $\forall \varepsilon>0$ $\exists N$: $\forall n>m>N$, $|x_n-x_m|<\varepsilon$




<u>Def</u>: $\mathbb{R}=\{\{x_n\}\text{ Cauchy}\}/\sim$

$\{x_n\}\sim\{y_n\} \iff x_n-y_n\to 0$.

---

<u>Def</u>: An <u>absolute value</u> on $\mathbb{Q}$ is $|\cdot|:\mathbb{Q}\to\mathbb{Q}$ s.t. $|x|\ge 0$, $|x|=0\Rightarrow x=0$, $|xy|=|x||y|$, $|x+y|\le|x|+|y|$. $\boxed{|\cdot|_1\sim|\cdot|_2 \iff \forall x,\ |x|_1<1\iff|x|_2<1}$

<u>Thm</u>: (Ostrowsky 1916): Classification of $|\cdot|$:

① $|\cdot|_\infty$ ② $|x|=\begin{cases}1 & \text{else}\\ 0 & x=0\end{cases}$ ③ $\left|p^n\frac{a}{b}\right|=p^{-n}$ $\boxed{(a,p)=1=(b,p)}$

Fix $p$.

(1)

$$\left|1+\frac{3}{17}\right|_5 = \left|\frac{20}{17}\right|_5 = \left|5\cdot\frac{4}{17}\right|_5 = \frac{1}{5}$$

---

$\mathbb{Q}_p = \{ \{x_n\}$ Cauchy seq under $|\cdot|_p \} / \sim$.

---

Back to $\mathbb{R}$, or $\mathbb{R}^d$, $|x| = (x_1^2 + \dots + x_d^2)^{1/2}$

$x = (x_1, \dots, x_d)$

$E \subset \mathbb{R}^d$: $E^c = \{ x \in \mathbb{R}^d \mid x \notin E \}$ complement.

---

$E \setminus F = E - F = \{ x \in E : x \notin F \}$

---

$$d(E, F) = \inf_{\substack{x \in E \\ y \in F}} d(x, y)$$

---

open ball: $B_r(x) = \{ y \in \mathbb{R}^d \mid d(x,y) < r \}$.

---

Def: $E$ is open: $\forall x \in E \ \exists r > 0$ s.t. $B_r(x) \subset E$.

$E$ is closed: $E^c$ open

~~Thm E~~ Def: $x$ is a limit pt (accumulation pt) of $E$

if $\forall B_r(x) \cap E \neq \emptyset$.

Thm: $E$ is closed $\iff$ Every limit pt of $E$ is in $E$.

pf. $\Rightarrow$ Assume $E$ closed, i.e. $E^c$ open.

~~Let $x$ be limit pt of~~ $E$, Assume $x \notin E$.

$\Rightarrow x \in E^c \Rightarrow \exists\, \varepsilon > 0 \ B_\varepsilon(x) \subset E^c$


$\Rightarrow B_\varepsilon(x) \cap E = \emptyset$

---

pf $\Leftarrow$) Assume $E$ contains all its limit pts.

Let $x \in E^c \Rightarrow$ $x$ not a limit pt of $E$

$\Rightarrow \exists\, \varepsilon > 0 : B_\varepsilon(x) \cap E = \emptyset. \Rightarrow B_\epsilon(x) \subset E^c \Rightarrow E^c$ open

---

Lemma: $\mathcal{O}_i$, $i \in \mathcal{I}$ open $\Rightarrow \bigcup \mathcal{O}_i$ open.

pf. $x \in \bigcup \mathcal{O}_i \Rightarrow x \in \mathcal{O}_i \Rightarrow B_\varepsilon(x) \subset \mathcal{O}_i \Rightarrow B_\epsilon(x) \subset \bigcup$

---

$$\bigcap_{n=1}^{\infty} \left(-\tfrac{1}{n}, 1 + \tfrac{1}{n}\right) = [0, 1].$$

---

Lemma: $\bigcap_{i=1}^{N} \mathcal{O}_i$ is open.

pf. $x \in \bigcap \mathcal{O}_i \Rightarrow x \in \mathcal{O}_i \ (\forall i) \Rightarrow \exists\, \varepsilon_i > 0 : B_{\varepsilon_i}(x) \subset \mathcal{O}_i$

$\Rightarrow B_{\varepsilon_{min}}(x) \subset \mathcal{O}_i \ \forall i$

---

Def. $E \subset \mathbb{R}^d$ is bdd. iff $\exists\, r > 0$: $E \subset B_r(x)$.
$\exists x \in \mathbb{R}^d$:

Def: $E$ is cpt (compact) $\Leftrightarrow$ Every open cover has a finite subcover

i.e. if $\bigcup_{i\in I} O_i \supset E \Rightarrow \exists\, I_0 \subset I,\ |I_0|<\infty$

s.t. $\bigcup_{i\in I_0} O_i \supset E$

---

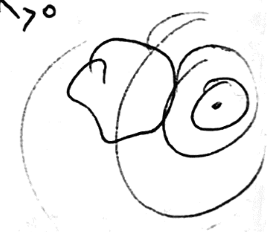
Dirichlet: (1852) Any cont function on ~~cpt~~ closed & bdd interval is uniformly continuous. implicitly assuming $[a,b]$ is cpt.

---

Thm (Heine-Borel) 1895: $E\subset \mathbb{R}^d$ cpt $\Leftrightarrow$ closed & bdd.

pf $\Rightarrow$ Say $E$ cpt, Claim: bdd. Let $\bigcup_{r\in\mathbb{R}_{>0}} B_r(0)\supset E$

& $\exists$ finite subcover, $\bigcup_{r_1,\dots,r_N} B_{r_i}(0)\supset E$.



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Say $E$ cpt: Claim $E$ is closed.

~~Let $x$ = limit pt of $E$.~~ Assume $x\notin E$.

$\bigcup_{r>0} \overline{B_r(x)}^c \supset E \Rightarrow \exists\ \overline{B_{r_1}(x)}^c \cup\dots\cup \overline{B_{r_N}(x)}^c \supset E$

$\Rightarrow \exists\ B_r(x)\cap E=\emptyset$. $\qquad \overline{B_{r_{\min}}(x)}^c \supset E$

pf $\Leftarrow$) Assume $E$ is closed & bdd. Claim: $E$ is cpt.

Step 1: $[-a,a]^d$ is cpt

Step 2: A closed subset of a cpt set is cpt.

pf(2): $K \subset E$, $K$ closed, $E$ cpt. Claim: $K$ cpt.

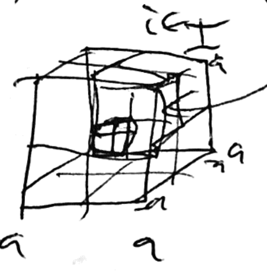
Let $\bigcup_{i\in I} O_i \supset K$. Then $\bigcup_{i\in I} O_i \cup K^c \supset E$

$E$ cpt $\Rightarrow \exists I_0$ finite $\bigcup_{i\in I_0} O_i \cup K^c \supset E$

$\Rightarrow \bigcup_{i\in I_0} O_i \supset K$

---

pf of (1): Assume $R_0=[-a,a]^d$ not cpt, $\exists \bigcup_{i\in I} O_i$ having no finite subcover

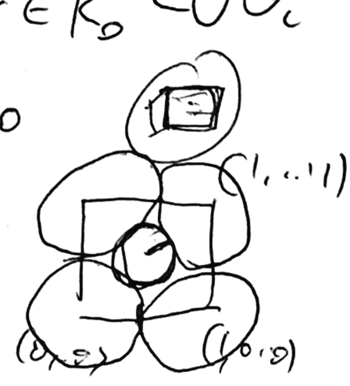



Let $R_0 = \bigcup_{l=1}^{2^d} R_1^{(l)} \subset \bigcup_{i\in I} O_i$

$\exists\, l_1$ s.t. $R_1^{(l_1)}$ has no finite subcover. Continue, getting $R_i^{(l_i)}$ each having no finite subcover. $\exists\, x_i \in R_i^{(l_i)}$

$\{x_i\}$ is Cauchy $\Rightarrow$ has limit pt $\Rightarrow x \in R_0 \subset \bigcup O_i$

$x \in O_i \Rightarrow \exists_{\varepsilon>0} B_\varepsilon(x) \subset O_i$, $\exists\, n \geq 0$ s.t. $\frac{1}{2^n} < \frac{1}{100\sqrt{d}}\varepsilon \Rightarrow R_0$ cpt.



**Def:** $x \in E$ is isolated: if $\exists\, r>0$ : $B_r(x) \cap E = \{x\}$.

**Def:** $x \in E$ is ~~an~~ an interior pt if: $\exists\, r>0$ : $B_r(x) \subset E$.

$E^\circ = \{x \in E : \text{interior}\}$ = ~~is~~ interior of $E$ = open.

$\overline{E} = E \cup \{\text{limit pts of } E\}$ = closure of $E$

**Lemma:** $\overline{E}$ is closed.

pf: Let $x$ be a limit pt of $\overline{E}$.


$\forall\, \varepsilon > 0$ ~~s~~, $B_\varepsilon(x) \cap \overline{E} \neq \emptyset$.

$\forall\, \frac{1}{n} > 0$, $B_{1/n}(x) \cap \overline{E} \neq \emptyset$, i.e. $B_{1/}$

$x_n \in \overline{E}$. if $x_n \notin E$, $\exists\, y_n \in B_{1/n}(x) \cap E$

$\Rightarrow x$ is limit pt of $E \Rightarrow x \in \overline{E}$.