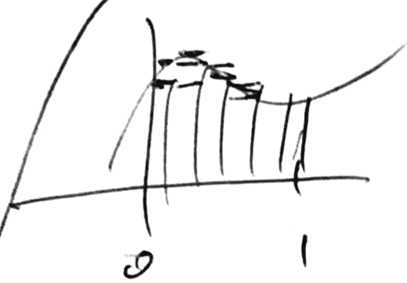
501 Real Analysis. Prof. Kontorovich.

$$f(x) = \begin{cases} 0 & \text{on } \mathbb{Q} \\ 1 & \text{on } \mathbb{R}/\mathbb{Q}. \end{cases}$$

Riemann: $\int_0^1 f(x)\,dx =$



1902 Lebesgue.

1820s Fourier. Solution to Heat Equ.

$u(x,t) =$ heat at x at time $t \geq 0$.

$u(0,t) = u(1,t) = 0, \quad u(x,0) = f(x).$

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2} u(x,t). \quad \text{PDE.}$$

Guess $u(x,t) = X(x)T(t)$.

$$X(x)T'(t) = X''(x)T(t).$$

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = c, \quad X(0) = X(1) = 0.$$

$$X'' - cX = 0.$$

$$X(x) = e^{\lambda x} \qquad X'' = \lambda^2 e^{\lambda x}$$

$$\lambda^2 e^{\lambda x} - c e^{\lambda x} = e^{\lambda x}(\lambda^2 - c) = 0.$$

$$\lambda = \pm\sqrt{c}.$$

$$X(x) = A e^{\sqrt{c}x} + B e^{-\sqrt{c}x} = A\left(e^{\sqrt{c}x} - e^{-\sqrt{c}x}\right).$$

$$0 = X(0) = A + B. \qquad B = -A.$$

$$0 = X(1) = A\left(\underbrace{e^{\sqrt{c}} - e^{-\sqrt{c}}}\right), \qquad e^{2\sqrt{c}} = 1.$$

$$e^{\sqrt{c}} = e^{-\sqrt{c}} \qquad c \neq 0.$$

$$2\sqrt{c} = 2\pi i n.$$

$$X(x) = A\left(e^{\pi i n x} - e^{-\pi i n x}\right).$$

$$\boxed{X(x) = A(\sin(\pi n x)).} \qquad c = -\pi^2 n^2.$$

$$n \in \mathbb{Z}.$$

$$\frac{T'(t)}{T(t)} = -\pi^2 n^2 \qquad C = -\pi^2 n^2.$$

$$\underline{T'(t) + \pi^2 n^2 T(t) = 0.}$$

$$T(t) = e^{\lambda t}, \quad T'(t) = \lambda e^{\lambda t}.$$

$$\lambda + \pi^2 n^2 = 0, \quad \lambda = -\pi^2 n^2.$$

$$T(t) = e^{-\pi^2 n^2 t}$$

$$X(x) = A \sin(\pi n x).$$

$$u(x,t) = \cancel{A} \sum_{n \geq 1}^{\infty} A_n \sin(\pi n x)\, e^{-\pi^2 n^2 t}$$

$$u(x,0) = f(x) = \sum_{n \geq 1}^{\infty} \underline{\underline{A_n}} \sin(\pi n x)$$

Fix m.

Fourier knows: $\int_0^1 \sin(\pi n x) \sin(\pi m x)\, dx = \frac{1}{2} \mathbb{1}_{n=m}$

$$\int_0^1 f(x) \sin(\pi m x)\, dx = \int_0^1 \left[\sum_{n \geq 1} A_n \sin(\pi n x) \right] \sin(m \pi x)\, dx$$

$$= \sum_{n \geq 1} A_n \left[\int_0^1 \sin(\pi n x) \sin(\pi m x)\, dx \right] = \frac{1}{2} \cdot A_m.$$

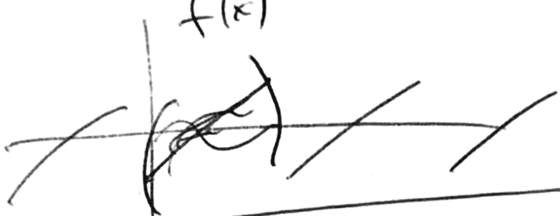
$$\boxed{u(x,t) = \sum_{n \geq 1} \left[2 \int_0^1 f(x) \sin(\pi n x)\, dx \right] \sin(\pi n x)\, e^{-\pi^2 n^2 t}}$$

Thm (Cauchy 1820s):

Bolzano 1817
Weierstrass 1860s.

Def: $u : \mathbb{R} \to \mathbb{R}$ is cont. at x if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $|h| < \delta$, $|u(x+h) - u(x)| < \varepsilon$.

If $S_n(x) \to S(x)$ & each S_n is continuous $\Rightarrow$ S is continuous.



pf: Given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall |h| < \delta$, $|S_n(x+h) - S_n(x)| < \varepsilon$

$\exists N : \forall n > N$, $|S_n(x) - S(x)| < \varepsilon$.

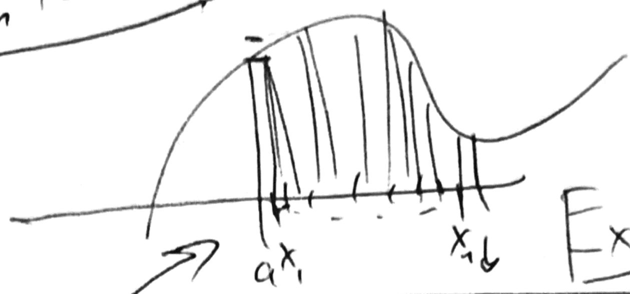
$|S_n(x+h) - S(x+h)| < \varepsilon$.

$$
\begin{aligned}
|S(x+h) - S(x)| \le\ & |S(x+h) - S_n(x+h)| \\
& + |S_n(x+h) - S_n(x)| \\
& + |S_n(x) - S(x)| \\
\le\ & 3\varepsilon.
\end{aligned}
$$

No!

Riemann 1854–1860s Thm: If f cont on $[a,b]$, then

$$\int_a^b f(x)\,dx \text{ exists.}$$

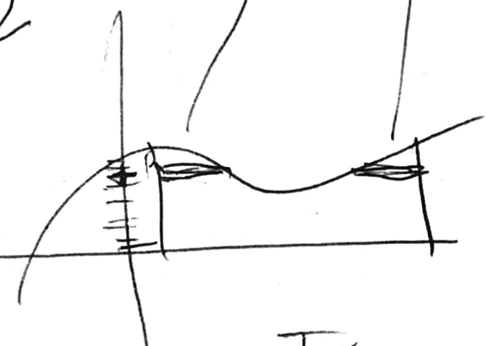


Exercise: prove this rigorously.

$P: a\le x_1 < \dots < x_n \le b$, $|P| = \max |x_{j+1}-x_j|$.

$$\lim_{|P|\to 0} \left[\sum_{i=1}^{n} \underline{f}(I_i)\,|I_i| - \sum_{i=1}^{n} \overline{f}(I_i)\,|I_i|\right] = 0. \qquad I_i = (x_i, x_{i+1}).$$

Lebesgue: 1902



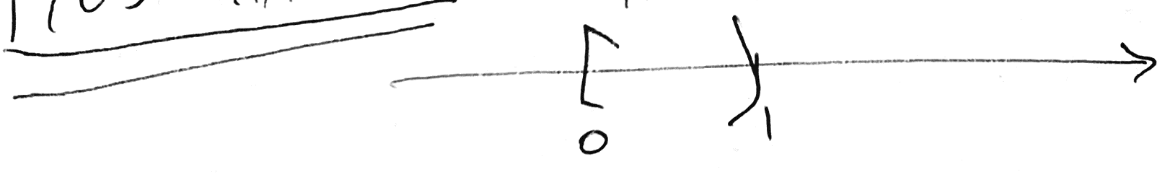
break y-axis, not x. Need measure theory.

Borel measure 1898.

Issues w/ measure theory.

1905 Vitali sets

$\mathbb{R}/\mathbb{Z} = \mathbb{R}/x\sim y$, $x \sim y \iff x-y = n\in\mathbb{Z}$



$N = \mathbb{R}/\mathbb{Q}$ $x\sim y \iff x-y\in\mathbb{Q}$.

Can choose rep's for $N \in [0,1)$.

Say $m(E \subset \mathbb{R})$ is measure function.

① $m(E+h) = m(E)$. (translation invariant)

② $m(\bigsqcup_i E_i) = \sum_i m(E_i)$ disjoint

③ $m([0,1]) = 1$.

$m(E) \ge 0$ or ∞.

$$\mathcal{N} = \mathbb{R}/\mathbb{Q} \subset [0,1)$$

for $r \in \mathbb{Q}\setminus 0$,

$$(\mathcal{N}+r) \cap \mathcal{N} = \emptyset$$

$$[0,1] \subset \underbrace{\bigsqcup_{-1 \le r \le 1} (\mathcal{N}+r)} \subset [-1,2).$$

$$1 \le m\Big(\bigsqcup_{-1 \le r \le 1} (\mathcal{N}+r)\Big) = \sum_{-1 \le r \le 1} \underbrace{m(\mathcal{N}+r)}_{= m(\mathcal{N})} \le 3.$$

The set $\mathcal{N}$ is <u>not measurable</u>

Why can we choose coset representatives????

Ax of Choice (Zermelo 1904).

Zermelo Frankel Axioms Set Th.

<u>Gödel + Cohen</u>: AC independent of ZF.