DE RHAM'S THEOREM

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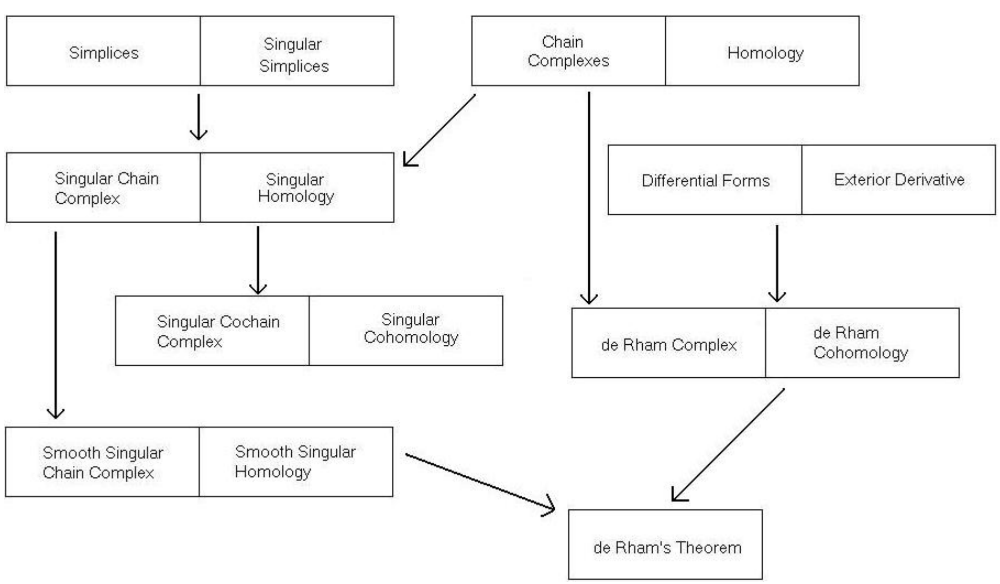
Contents

1. Introduction

When dealing with a smooth manifold, how can we use the smooth structure to detect topological properties? In 1931, the Swiss mathematician Georges de Rham proved that we can use the differentiable structure of smooth manifolds to recover the topological data detected by singular homology. The general hope of these notes is to be a relatively self-contained (if the reader has had a little differential geometry and topology) proof of de Rham's theorem. The general plan of these notes is as follows (description follows the figure.).



We will briefly define and describe singular homology and cohomology of topological spaces. Afterwards, we introduce a smooth variant called smooth singular homology and show that it is naturally isomorphic to singular homology. Finally, we present de Rham cohomology and prove de Rham's theorem, which tells us that singular cohomology and de Rham cohomology are naturally isomorphic.

2. Singular Homology

We begin by presenting a concept from homological algebra that is ubiquitous in modern mathematics and will prove indispensable in our constructions.

Definition 2.1. Let R be a commutative ring and C_n, $n \in \mathbb{Z}$, be R-modules along with R-homomorphisms

$$\partial_n : C_n \to C_{n-1}$$

for $n \in \mathbb{Z}$ satisfying $\partial_{n-1} \circ \partial_n = 0$. Then

$$\mathcal{C}: \quad \cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

is called a **chain complex** of R-modules. Elements of C_n are called n**-chains** or simply **chains**. Where there is no confusion, we write ∂ without the subscript for simplicity. We call the maps ∂ **boundary operators** or **differential operators**.

The fact that the maps are all going down with respect to the indices is purely formal (since simply negating the indices would make them 'go up'). For our purposes, we will deal mainly with abelian groups ($\mathbb{Z}$-modules) and real vector spaces ($\mathbb{R}$-modules).

Definition 2.2. Let $\mathcal{C}$, $\mathcal{C}'$ be chain complexes with modules C_n, C_n' (resp.) and boundary operators ∂, ∂' (resp.), then a **chain map** $s : \mathcal{C} \to \mathcal{C}'$ is a family of homomorphisms $s_n : C_n \to C_n'$ that commute with the boundary operators, i.e., such that $s_{n-1}\partial_n = \partial_n' s_n$.

Definition 2.3. Let R be a commutative ring and

$$\mathcal{C}: \ \cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

be a chain complex of R-modules, then the quotient modules

$$H_n = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$$

are called the n**th homology modules** of $\mathcal{C}$.

Inspired by geometry, the elements of $\ker \partial_n$ are called **cycles** and the elements of $\operatorname{im} \partial_{n+1}$ are called **boundaries**. We also write $Z_n = \ker \partial_n$ and $B_n = \operatorname{im} \partial_{n+1}$.

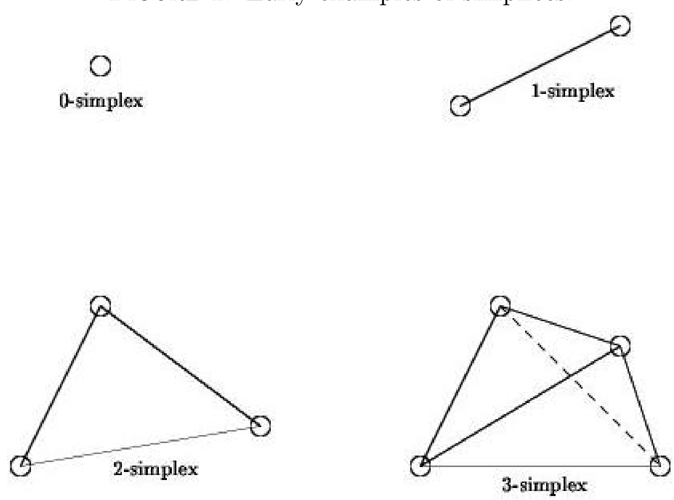
Now we specialize to a very useful construction for topological spaces. A very broad outline of the philosophy is as follows. Suppose that X is a (2 dimensional) surface and we would like to know how many 'holes' it has. To find out if X has a hole, we can draw a closed curve, i.e., we can find a continuous map $\sigma : S^1 \to X$, and ask if the closed curve can be 'filled in', i.e., ask if this map has a continuous extension $\tilde{\sigma} : D \to X$ where D is the closed disk. If there is a hole or an obstruction of some sort then this map has no extension, i.e., no 'filling in' is possible. Furthermore, instead of using circles and disks we can use triangles and solid triangles to the same effect (with the added benefit that a solid triangle can be easily subdividing into smaller solid triangles, unlike disks). Another way of asking the above question, while employing the language we introduced, is to ask: which cycles (or closed loops) are actually boundaries (of higher dimensional contractible spaces)? With this approach in mind, we generalize the concept of solid triangle to any dimension and use them to find data about topological spaces.

Definition 2.4. Let $v_0, v_1, \ldots, v_k$ be $k{+}1$ many points in $\mathbb{R}^n$ that are *not* contained in a subspace of dimension $k-1$, then the convex hull of these points

$$s = \left\{ \sum_{j=0}^{k} t_j v_j : \sum_{j=0}^{k} t_j = 1,\ t_j \geq 0 \right\}$$

is called a **k-simplex**. If the points come with an ordering then s is called an **ordered k-simplex** and is denoted $[v_0, \ldots, v_k]$. The points $v_0, v_1, \ldots, v_k$ are called the **vertices** of $[v_0, \ldots, v_k]$ and are uniquely determined by the k-simplex (up to ordering). The **ith face** (or simply a **face** if unordered) of $[v_0, v_1, \ldots, v_k]$ is the $(k-1)$-simplex $[v_0, \ldots, \hat{v}_i, \ldots, v_k]$.

FIGURE 1. Early examples of simplices.



Definition 2.5. Let $e_0, e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^{n+1}$ then $[e_0, e_1, \ldots, e_n]$ is called the **standard n-simplex**. It is also denoted by s_n.

Definition 2.6. Let X be a topological space, then a continuous map $\sigma : s_n \to X$ is called a **singular n-simplex**. The **ith face of σ**

$$\sigma|[e_0, \ldots, \hat{e}_i, \ldots, e_n] : s_{n-1} \to X$$

is understood to be the composite

$$s_{n-1} \to [e_0, \ldots, \hat{e}_i, \ldots, e_n] \hookrightarrow [e_0, \ldots, \ldots, e_n] \to X$$

where the first map, which is called the **ith face map** denoted by F_i (when n is understood), is the canonical identification of simplices that preserves the order of the vertices. (Note that the notation is similar to our notation of a restriction map (but it is not exactly a restriction map!).) For simplicity we write $\sigma_i = \sigma|[e_0, \dots, \hat{e_i}, \dots, e_n]$.

Why the term 'singular'? This is to emphasize the fact that the maps $\sigma : s_n \to X$ do not have to be differentiable or smooth in any way. It is rather surprising that including so many, possibly pathological, maps gives good information even in the case of smooth manifolds. In a later section, we show that restricting our attention to smooth maps results in no loss of data whatsoever. The next proposition shows how we organize the data contained in the set of all singular simplices and the reader is encouraged to prove it him or herself.

Proposition 2.7. *Let X be a topological space and let $C_n(X)$ be the free abelian group generated by the set of all singular n-simplices for $n = 0, 1, 2, \dots$ and consider the homomorphism, which we define on the bases,*

$$\partial_n : C_n(X) \to C_{n-1}(X)$$
$$\partial_n \sigma = \sum_{j=0}^{n} (-1)^j \sigma_j = \sum_{j=0}^{n} (-1)^j \sigma \circ F_j.$$

Then

$$\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots$$

(where the negative indexed groups and maps are defined to be 0*) is a chain complex of abelian groups.*

Definition 2.8. Let X be a topological space, then the above chain complex is called the **singular chain complex of** X and the homology groups

$$H_n(X) = \frac{\ker \partial_n}{\mathrm{im}\ \partial_{n+1}}$$

are called the **singular homology groups of** X.

A crucial feature of singular homology is that it is homotopy invariant. We will not provide a proof of this here but record it and another important concept here.

Definition 2.9. Let $\mathcal{C}, \mathcal{C}'$ be chain complexes with boundary operators ∂, ∂'. Two chain maps s, s' from $\mathcal{C}$ to $\mathcal{C}'$ are called **chain homotopic** if there is a collection of maps $\{h_n : C_n \to C'_{n+1}\}_{n \in \mathbb{Z}}$ so that

$$\partial' h_n + h_{n-1} \partial = s - s'.$$

Theorem 2.10. *Let X and Y be topological spaces, $f : X \to Y$, $g : X \to Y$ be homotopic continuous maps, then f and g induce the same homomorphisms on homology. More precisely, the homomorphisms induced by f and g between singular homologies are chain homotopic.*

In particular, if two spaces X and Y are homotopy equivalent, then they have isomorphic homology groups. We stress the importance of this fact for two reasons. The first is that homology is defined independently of homotopy and so it is a small miracle that they work so well together. The second is that when complicated spaces are homotopy equivalent to simpler ones, we are allowed to compute the homology of the simpler one, e.g., contractible spaces (spaces homotopy equivalent to a point) have the same singular homology as that of a point.

Exercise 2.11. Show that

$$H_n(\{x\}) \cong \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n > 0 \end{cases}$$

3. Singular Cohomology

Dual to the concept of singular homology is singular cohomology, which considers the dual space of chain complexes and asks the same question with respect to the (induced) boundary operators.

Definition 3.1. Let X be a topological space with singular chain complex

$$\mathcal{C}: \ \cdots \to C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} \cdots$$

and G be an abelian group, then the chain complex

$$\mathcal{C}^*: \ \cdots \leftarrow C^*_{n+1}(X;G) \xleftarrow{\delta} C^*_n(X;G) \xleftarrow{\delta} C^*_{n-1}(X;G) \xleftarrow{\delta} \cdots,$$

where $C^*_n(X;G) = \hom(C_n(X), G)$ and $\delta = \partial^*$ is the map $\delta(\lambda)(\sigma) = \lambda(\partial\sigma)$, is called the **singular cochain complex of X with coefficients in G**. The elements of C^*_n are called **n-cochains** or simply **cochains**.

Proposition 3.2. *$\mathcal{C}^*$ is a chain complex.*

Proof. The $C^*_n(X;G)$ are $\mathbb{Z}$-modules, so we need only check that $\delta\delta = 0$ and indeed we have

$$\delta\delta(\lambda)(\sigma) = \delta(\lambda)(\partial\sigma) = \lambda(\partial\partial\sigma) = 0.$$

□

Definition 3.3. Let X be a topological space with singular cochain complex $\mathcal{C}^*$ then the homology groups of $\mathcal{C}^*$

$$H^n(X;G) = \frac{\ker(\delta : C^*_n(X;G) \to C^*_{n+1}(X;G))}{\mathrm{im}\ (\delta : C^*_{n-1}(X;G) \to C^*_n(X;G))}$$

are called the **singular cohomology groups of X with coefficients in G**.

Analogously with homology, the elements of $\ker\delta$ are called **cocycles** and the elements of $\mathrm{im}\,\delta$ are called **coboundaries**. There are two very important properties of cohomology that we will need to prove de Rham's theorem. The first is homotopy invariance, which we summarize by saying that homotopy equivalent spaces have isomorphic cohomology groups, just as with singular homology.

The second important property of singular cohomology (and of singular homology) is its behavior with respect to an open cover consisting of exactly two open sets. Specifically, for any topological space X with open cover $\{U,V\}$ it related the cohomologies of X, U, V, and $U \cap V$ as described below.

Theorem 3.4 (Mayer-Vietoris)**.** *Let X be a topological space, G be an abelian group, and $\{U,V\}$ be an open cover of X. For every n, there is a map*

$$\Delta : H^{n-1}(U \cap V; G) \to H^n(X;G)$$

such that the following sequence is exact:

$$\cdots \longrightarrow H^{n-1}(U;G) \oplus H^{n-1}(V;G) \longrightarrow H^{n-1}(U \cap V;G) \xrightarrow{\Delta}$$

$$H^n(X;G) \longrightarrow H^n(U;G) \oplus H^n(V;G) \longrightarrow \cdots$$

It is natural to ask if taking the dual of homology groups of a chain complex is the same thing as taking the homology of the dual of a chain complex, i.e., if $\hom(H_n(X), G)$ and $H^n(X;G)$ are the same. In general, the answer is no and this is expressed in the so-called *universal coefficient theorem for cohomology.* As a special case, when $G = \mathbb{R}$, the fact that $\mathbb{R}$ is an injective group makes $H^n(X;G) \cong \hom(H_n(X), G)$ via the natural map $[\lambda] \mapsto ([c] \mapsto \lambda(c))$. Again, we will not delve deeper into this fact here since it will take us too far afield.

4. Smooth Singular Homology

Passing to the category of smooth manifolds, we would like to find a smooth variant to singular homology. It has the somewhat contradictory name *smooth singular homology.* First, recall that an n-simplex is a manifold with corners and a smooth singular n-simplex is a map that is smooth on some extension to a neighborhood of every point (in some chart).

Definition 4.1. Let M be a smooth manifold and $\mathcal{C}^\infty$ be the chain complex

$$\cdots \to C^\infty_{n+1}(M) \xrightarrow{\partial} C^\infty_n(M) \xrightarrow{\partial} C^\infty_{n-1}(M) \xrightarrow{\partial} \cdots$$

where $C^\infty_n(M)$ is the free abelian subgroup of $C_n(M)$ generated by the set of all *smooth* singular n-simplices and ∂ is the corresponding restriction map. Then $\mathcal{C}^\infty$ is called the **smooth singular chain complex of** M and

$$H^\infty_n(M) = \frac{\ker\left(\partial : C^\infty_n(M) \to C^\infty_{n-1}(M)\right)}{\operatorname{im}\left(\partial : C^\infty_{n+1}(M) \to C^\infty_n(M)\right)}$$

is called the nth **smooth singular homology group of** M. The elements of $C^\infty_n(M)$ are called **smooth n-chains**.

Remark 4.2. Do not confuse these with the space of smooth functions on M!

We may sometimes omit the ‘singular’ part of terms such as ‘smooth singular homology’ for simplicity. The following proposition is straightforward and the proof is left to the reader.

Proposition 4.3. *The above definition makes sense, i.e., $\partial(C^\infty_n(M)) \subseteq C^\infty_{n-1}(M)$.*

We can define smooth singular cohomology the same way. What is the difference between singular homology and smooth singular homology? Fortunately, there is none.

Theorem 4.4. *Let M be a smooth manifold, then the homomorphism $i_* : H^\infty_n(M) \to H_n(M)$ induced by the inclusion map $i : C^\infty_n(M) \to C_n(M)$ is an isomorphism.*

Proof. To prove this, we will construct a chain map $j : \mathcal{C}(M) \to \mathcal{C}^\infty(M)$ that is a left inverse to the inclusion map $i : \mathcal{C}^\infty(M) \to \mathcal{C}(M)$ and so that $i \circ j$ is chain homotopic to identity. To produce j we will first show that singular simplices are homotopic to smooth singular simplices (in a consistent way). More precisely, for every singular n-simplex σ we will show the existence of a homotopy $H_\sigma : s_n \times I \to M$ satisfying:

1. (Homotopy with smooth simplex) $H_\sigma(x, 0) = \sigma(x)$ and $H_\sigma(x, 1)$ is a smooth singular n-simplex.
2. (Compatibility on faces) $H_{\sigma_i}(x, t) = H_\sigma(F_i, t)$ where $F_i : s_{n-1} \to s_n$ and $\sigma_i = \sigma \circ F_i$.
3. (Constant on smooth simplices) If σ is smooth then $H_\sigma(x, t) = \sigma(x)$ for all $t \in I$.

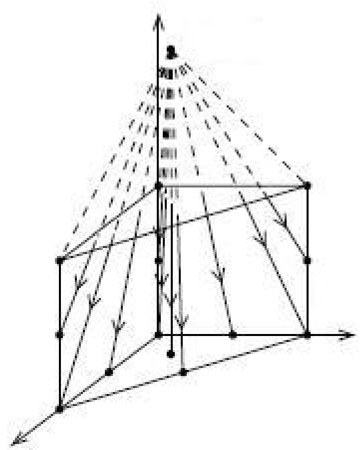
We proceed by induction on n. When $n = 0$, the constant homotopy does the trick. Now, suppose that we have proved it for all $k < n$ and let σ be a singular n-simplex. If we set

$$X = (s_n \times \{0\}) \cup (\partial s_n \times I)$$

and define

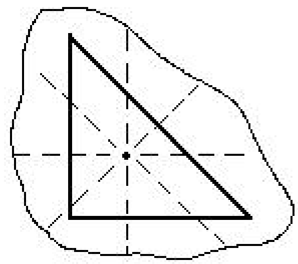
$$G(x,t) = \begin{cases} \sigma(x), & t = 0 \\ H_{\sigma_i}(x,t) & x \in \partial s_n \in I,\ t \in I \end{cases}$$

This map is continuous since by induction $H_{\sigma_i}(x, 0) = \sigma_i(x) = \sigma(x)$. We extend this to a homotopy $H : s_n \times I \to M$ by introducing a retraction $r : s_n \times I \to X$ and composing $H_1 = G \circ r$. An example of such a retraction is illustrated in the figure below.



Now, we cannot be sure that $H_1(x, 1)$ gives a smooth simplex. To fix this, consider the natural diffeomorphism from the n-simplex $[0, e_1, \ldots, e_n] \subseteq \mathbb{R}^n$ to s_n that sends vertices to vertices in the same order. This gives a map $H(\cdot, 1) = \sigma' : [0, e_1, \ldots, e_n] \to M$ which has an extension $\tilde{\sigma}'$ to an open set $U \supseteq [0, e_1, \ldots, e_n]$ via the continuous 'radial map' sending every element outside of $[0, \ldots, e_n]$ to the boundary along lines emanating from an interior point of the simplex, illustrated in the figure below.

We claim that σ' restricted to the boundary is smooth (see [Lee] pg. 477-478) and so we can invoke the Whitney approximation theorem which tells us that



there is a homotopy $H_2 : U \times I \to M$ rel $\partial[0,\dots,e_n]$ (recall that rel means that $H_2(\cdot,t)|_{\partial[0,\dots,e_n]} \equiv H_2(\cdot,0)$ for all $t \in I$) so that $H_2(\cdot,0) = \tilde{\sigma}'$ and $H_2(\cdot,1)$ is smooth.

It remains to put these two homotopies together in such a way as to respect the compatibility on faces property. To this end, we consider a function $u : s_n \to \mathbb{R}$ such that $u|_{\partial s_n} \equiv 1$ and $0 < u|_{Int(s_n)} < 1$ and then define

$$H_\sigma(x,t) = \left\{ \begin{array}{ll} H_1\left(x, \frac{t}{u(x)}\right), & x \in s_n,\ 0 \le t \le u(x) \\ H_2\left(x, \frac{t-u(x)}{1-u(x)}\right) & x \in Int(s_n),\ u(x) \le t \le 1 \end{array} \right. .$$

We claim that this is the desired homotopy (check continuity!). We define

$$j(\sigma)(x) = H_\sigma(x,1)$$

and extend j linearly to all of $C_n(M)$, which satisfies the first property, i.e., it is a left inverse of the inclusion map $i : C_n^\infty(M) \to C_n(M)$, by the fact that H_σ is constant on smooth simplices.

To show that $i \circ j$ is chain homotopic to identity, we introduce homomorphisms $h : C_n(M) \to C_{n+1}(M)$ that are very similar to the so-called *prism operators* used to show that homotopic maps f, g induce chain homotopic maps on chain complexes (for further discussion on these, see [Hat] pg. 111-112). The basic idea is to subdivide $s_n \times I \subseteq \mathbb{R}^n \times \mathbb{R}$ into $(n+1)$-simplices. Let $s_n \times \{0\} = [v_0, v_1, \dots, v_n]$ and $s_n \times \{1\} = [w_0, w_1, \dots, w_n]$ where $w_i = v_i + (0,\dots,0,1)$ and consider the natural order preserving homeomorphisms

$$\tau_0 : s_{n+1} \to [v_0, w_0, w_1, \dots, w_n],$$

$$\tau_1 : s_{n+1} \to [v_0, v_1, w_1, w_2, \dots, w_n],$$

$$\vdots$$

$$\tau_n : s_{n+1} \to [v_0, v_1, \dots, v_n, w_n].$$

Using this data we define the map

$$h(\sigma) = \sum_{i=0}^{n} (-1)^i H_\sigma \circ \tau_i.$$

Letting F_j represent the jth face map, i.e., the jth face is $\sigma_j = \sigma \circ F_j$, notice that

$$(F_j \times Id) \circ \tau_i = \left\{ \begin{array}{ll} \tau_{i+1} \circ F_j, & i \ge j \\ \tau_i \circ F_{j+1}, & i < j \end{array} \right.$$

and that $\tau_0 \circ F_0 = [w_0, \dots, w_n]$, $\tau_n \circ F_{n+1} = [v_0, \dots, v_n]$. Now we compute

$$\begin{aligned}
h(\partial\sigma) &= h \sum_{j=0}^{n} (-1)^j \sigma_j \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} H_{\sigma_j} \circ \tau_i \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} H_{\sigma} \circ (F_j \times Id) \circ \tau_i \\
&= \sum_{0 \le j \le i \le n-1} (-1)^{i+j} H_\sigma \circ \tau_{i+1} \circ F_j \\
&+ \sum_{0 \le i < j \le n} (-1)^{i+j} H_\sigma \circ \tau_i \circ F_{j+1},
\end{aligned}$$

$$\begin{aligned}
\partial(h\sigma) &= \partial \sum_{i=0}^{n} (-1)^i H_\sigma \circ \tau_i \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} H_\sigma \circ \tau_i \circ F_j
\end{aligned}$$

One can check that

$$h(\partial\sigma) + \partial(h\sigma) = H_\sigma \circ \tau_0 \circ F_0 - H_\sigma \circ \tau_n \circ F_{n+1} = j(\sigma) - \sigma$$

and since $ij = j$, we have proved the result. □

Via this isomorphism, we can identify singular cohomology with coefficients in $\mathbb{R}$ with its smooth counterpart given that $H^n(X; \mathbb{R}) = \hom(H_n(X), \mathbb{R})$.

5. de Rham Cohomology

We briefly introduce de Rham cohomology. We begin by introducing the de Rham complex which gives a natural way of organizing the space of differential forms on a smooth manifold M.

Definition 5.1. Let M be a smooth manifold and $\Omega^k(M)$ be the real vector space of differential k-forms on M for $k = 0, 1, 2, \dots$, then

$$0 \to \Omega_0(M) \xrightarrow{d} \Omega_1(M) \xrightarrow{d} \Omega_2(M) \xrightarrow{d} \cdots$$

is a chain complex, where d is the exterior derivative. The homology group (or cohomology groups)

$$H^k_{dR}(M) = \frac{\ker(d \colon \Omega_n(M) \to \Omega_{n+1}(M))}{\operatorname{im}(d \colon \Omega_{n-1}(M) \to \Omega_n(M))}$$

is called the **kth de Rham cohomology group of M**.

Elements of the kernel $\ker(d \colon \Omega_n(M) \to \Omega_{n+1}(M))$ are called **closed n-forms** while elements of $\operatorname{im}(d \colon \Omega_{n-1}(M) \to \Omega_n(M))$ are called **exact n-forms**. Recall that if M, N are smooth manifolds and $f : M \to N$ is a smooth map, then for any k-form ω on N, the k-form $f^*\omega$ on M is called the **pullback of** ω. This map commutes with the exterior derivative d and so induces a map on cohomology, which we also denote f^*,

$$f^* : H^k_{dR}(N) \to H^k_{dR}(M)$$

which is called the **induced cohomology map**.

We record some of the more important facts regarding de Rham cohomology without proof.

Proposition 5.2. *Let M be a smooth manifold (with or without boundary) of dimension n, then $H^m_{dR}(M) = 0$ for $m > n$ and $m < 0$.*

Proposition 5.3. *Homotopy equivalent smooth manifolds have isomorphic de Rham cohomology.*

Proposition 5.4. *Let $\{M_j\}_j$ be a countable collection of smooth manifolds (with or without boundary), then for all n,*

$$H^n_{dR}\left(\coprod_j M_j\right) \to \prod_j H^n_{dR}(M_j)$$

with the maps induced by the inclusion maps $i_j : M_j \to \coprod_i M_i$.

Proposition 5.5. *Let M be a connected smooth manifold (with or without boundary), then $H^0_{dR}(M) \cong \mathbb{R}$.*

The de Rham cohmology groups also have a Mayer-Vietoris sequence.

Theorem 5.6 (Mayer-Vietoris)**.** *Let M be a smooth manifold (with or without boundary) and let $\{U, V\}$ be an open cover of M. For every n, there is a map $\Delta : H^{n-1}_{dR}(U \cap V) \to H^n_{dR}(M)$ such that the following sequence is exact:*

$$\cdots \longrightarrow H^{n-1}_{dR}(U) \oplus H^{n-1}_{dR}(V) \longrightarrow H^{n-1}_{dR}(U \cap V) \xrightarrow{\Delta} H^n_{dR}(M) \longrightarrow$$

$$H^n_{dR}(U) \oplus H^n_{dR}(V) \longrightarrow H^n_{dR}(U \cap V) \longrightarrow \cdots$$

6. de Rham's Theorem

Now, we present de Rham's theorem, which states that de Rham cohomology is naturally isomorphic to singular cohomology and that the isomorphism is given by integration. Integrating smooth simplices is defined as

$$\int_\sigma \omega := \int_{[e_0,\dots,e_n]} \sigma^*\omega$$

and for a smooth chain c we extend this definition linearly. Before presenting the theorem, we must show that integration elements of smooth singular homology makes sense.

Theorem 6.1 (Stokes' Theorem)**.** *If c is a smooth k-chain in a smooth manifold M and ω is a smooth $(k-1)$-form on M, then*

$$\int_{\partial c} \omega = \int_c d\omega.$$

In particular, if two smooth cycles define the same homology class then integrating them against ω gives the same result.

Proof. By the linearity of integration, we can verify the theorem when $c = \sigma$ is a smooth k-simplex. Now, s_k, interpreted here as $[0 = e_0, e_1, \dots, e_k] \subseteq \mathbb{R}^k$ is a smooth manifold with corners, so the standard Stokes' theorem tells us that

$$\int\limits_{\sigma} d\omega = \int\limits_{s_k} \sigma^* d\omega = \int\limits_{s_k} d\sigma^*\omega = \int\limits_{\partial s_k} \sigma^*\omega,$$

where ∂s_k has the Stokes' orientation (induced by the outward pointing normal). Note that F_j is the restriction to $s_k \cap \partial\mathbb{H}^k$ of the natural diffeomorphism sending the simplex $[0 = e_0, e_1, \dots, e_k]$ to $[e_0, \dots, \hat{e_i}, \dots, e_k, e_i]$, which is orientation preserving if and only if $(e_0, \dots, \hat{e_i}, \dots, e_k, e_i)$ is an even permutation of $(e_0, e_1, \dots, e_k)$, i.e., if $k-i$ is even. Since the standard orientation of $\partial\mathbb{H}^k$ matches the standard orientation of $\mathbb{R}^k$ when k is even and is its negative when k is odd, it follows that F_j is orientation preserving if and only if j is even. But then we get that

$$\int\limits_{\partial s_k} \sigma^*\omega = \sum_{j=0}^{k} (-1)^j \int\limits_{s_{k-1}} F_j^*\sigma^*\omega = \sum_{j=0}^{k} (-1)^j \int\limits_{s_{k-1}} (\sigma \circ F_j)^*\omega = \int\limits_{\partial\sigma} \omega$$

as desired. □

It is a good idea to write down the explicit maps described above if seeing this for the first time. This gives a map $H^k_{dR}(M) \to \hom(H_n(M); \mathbb{R})$ and since the right side is naturally isomorphic to $H^k(M;\mathbb{R})$. This means that integrating against a closed differential form does not distinguish between cycles in the same homology class. This leads us to the theorem that we came here for.

Theorem 6.2. *The homomorphism*

$$\mathcal{I} \colon H^k_{dR}(M) \to H^k(M;\mathbb{R})$$

$$\mathcal{I}([\omega])([c]) = \int\limits_{c} \omega,$$

where c is a smooth *k-cycle, is an isomorphism for all $k \geq 0$.*

Proof. For the purposes of this proof, we say that a smooth manifold M is a **de Rham manifold** if $\mathcal{I} \colon H^k_{dR}(M) \to H^k(M;\mathbb{R})$ is an isomorphism. The theorem can therefore be restated: Every smooth manifold is a de Rham manifold. Furthermore, for any smooth manifold M, an open cover $\{U_i\}_i$ is called a **de Rham cover** if every finite intersection of the U_i is de Rham. If the open cover is also a topological basis, then it is called a **de Rham basis**.

First, notice that the map $\mathcal{I}$ is natural in the sense that for any smooth map $F \colon M \to N$ the diagram

$$\begin{array}{ccc} H^n_{dR}(N) & \xrightarrow{F^*} & H^n_{dR}(M) \\ \downarrow{\scriptstyle \mathcal{I}} & & \downarrow{\scriptstyle \mathcal{I}} \\ H^n(N;\mathbb{R}) & \xrightarrow{F^*} & H^n(M;\mathbb{R}) \end{array}$$

commutes since for any smooth chain c, $\int\limits_c F^*\omega = \int\limits_{F(c)} \omega$. Given this fact, the main case to consider in the proof is when a smooth manifold M has a *finite* de Rham cover. To show that such an M is itself de Rham, we proceed by induction on the size of the open cover. If M is covered by a single de Rham open set, then we are done. Now, if $M = U \cup V$ where U, V are open and $U \cap V$ is de Rham, then the Mayer-Vietoris sequence

$$\begin{tikzcd}
H^{n-1}_{dR}(U) \oplus H^{n-1}_{dR}(V) \arrow[r] \arrow[d, "\mathcal{I}"'] & H^{n-1}_{dR}(U \cap V) \arrow[r] \arrow[d, "\mathcal{I}"'] & H^{n}_{dR}(M) \arrow[r] \arrow[d, "\mathcal{I}"'] & {} \\
H^{n-1}(U;\mathbb{R}) \oplus H^{n-1}(V;\mathbb{R}) \arrow[r] & H^{n-1}(U \cap V;\mathbb{R}) \arrow[r] & H^{n}(M;\mathbb{R}) \arrow[r] & {}
\end{tikzcd},$$

$$\begin{tikzcd}
H^{n}_{dR}(U) \oplus H^{n}_{dR}(V) \arrow[r] \arrow[d, "\mathcal{I}"'] & H^{n}_{dR}(U \cap V) \arrow[r] \arrow[d, "\mathcal{I}"'] & {} \\
H^{n}(U;\mathbb{R}) \oplus H^{n}(V;\mathbb{R}) \arrow[r] & H^{n}(U \cap V;\mathbb{R}) \arrow[r] & {}
\end{tikzcd}$$

the naturality of $\mathcal{I}$, and the five lemma show that $\mathcal{I}$ gives an isomorphism in the middle. For the general inductive step, let $U_1, \dots, U_{k+1}$ be a de Rham cover of M for $k \geq 2$, and write $U = U_1 \cup \cdots \cup U_k$, $V = U_{k+1}$. Since $U_1 \cap V, \dots, U_k \cap V$ is a de Rham cover of U (by the inductive hypothesis), we know that U, V is a de Rham cover of M. It follows from the above work that M is de Rham.

For the general case, we would like to show that a basis of open coordinate balls of a smooth manifold M constitute a de Rham basis and that manifolds with de Rham bases are themselves de Rham. For the first assertion, any open ball B in $\mathbb{R}^n$ is a convex set and so is contractible. Due to the contractibility, all positive dimensional cohomology spaces vanish (for both de Rham and singular). At the 0-dimensional level, the integration map $\mathcal{I}$ applied to a constant function gives a nonzero element of $H^0(\mathrm{B}) \cong \mathbb{R}$ and hence an isomorphism. Note that similar reasoning shows that a collection of open balls is a de Rham cover.

For the second assertion, we cleverly use an exhaustion function $f\colon M \to \mathbb{R}$ and consider the annuli

$$A_m = f^{-1}([m, m+1]), \quad A'_m = f^{-1}\left(\left(m - \frac{1}{2}, m + \frac{3}{2}\right)\right).$$

Since the A_m are compact and $A_m \subseteq A'_m$, it follows that we can find a finite open cover of A_m consisting of basis sets that lie in A'_m. Let B_m be the union of these and notice that $A_m \subseteq B_m \subseteq A'_m$. The B_m are de Rham because they have a finite de Rham cover. By construction, A'_i, A'_j intersect for distinct i, j if and only if i, j are consecutive integers. If we set

$$U = \bigcup_{i\ odd} B_i, \quad V = \bigcup_{i\ even} B_i$$

then U and V are disjoint unions of countably many de Rham manifolds. Since for any countable collection of de Rham manifolds M_j, the inclusion maps $M_i \hookrightarrow$

$\coprod_j M_j$ induce isomorphisms

$$H^n_{dR}\left(\coprod_j M_j\right) \to \prod_j H^n_{dR}(M_j)$$
$$H^n\left(\coprod_j M_j; \mathbb{R}\right) \to \prod_j H^n(M_j; \mathbb{R}),$$

the naturality of $\mathcal{I}$ gives an isomorphism of the left terms above, i.e., they show that the countable disjoint union of de Rham manifolds is de Rham. Now that we have established that U and V are de Rham open sets that cover M, notice that $U \cap V$ is the disjoint union of $B_i \cap B_{i+1}$, which have a finite de Rham cover (consisting of pairwise intersections of the basis sets covering B_i and B_{i+1}) by construction. It follows that $\{U, V\}$ is a finite de Rham cover of M, in turn showing that M is de Rham.

The theorem follows from the fact that every smooth manifold has a basis consisting of open coordinate balls. □

Notice that in the course of the proof we showed that the isomorphism is a *natural* one. This wonderful proof is due to Glen E. Bredon and can be found in [Lee].

7. References

[Lee] Lee, John M., Introduction to Smooth Manifolds, second edition, Springer Science+Business Media New York, 2013.

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