Notes on Vertex Operator Algebras and their Representations (in progress)

Contents

1 Vertex Operator Algebras

1.1 Introduction

Classical mathematics was often concerned with finite dimensional algebraic objects: vector spaces, associative algebras, Lie algebras, etc. As interest in infinite dimensional objects increased for both mathematical and physical reasons, we sought to find well-behaved classes of such objects for study. One fruitful direction that features prominently in the investigation of the Moonshine conjectures, affine Lie algebras, Heisenberg algebras, Virasoro algebras, conformal field theory, and a growing list of topics is the study of *vertex operator algebras*. Here we present some notes on this fascinating subject.

1.2 Definition of Vertex Operator Algebra

Throughout, we will work over a field $\mathbb{F}$ of characteristic 0. Define the *binomial coefficients* to be

$$\binom{n}{k} := \frac{n \cdot (n-1) \cdots (n-k+1)}{k!}$$

for $k > 0$, $n \in \mathbb{Z}$, and

$$\binom{n}{0} := \delta_{n,0}$$

for $n \in \mathbb{Z}$. The space of formal power series $\mathbb{F}[[x_1, x_2, \dots, x_n]]$ is the vector space consisting of elements of the form

$$\sum_{\alpha \in \mathbb{Z}^n} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

where $c_\alpha \in \mathbb{F}$. For any vector space V obver $\mathbb{F}$, we define $V[[x_1, \dots, x_n]] := V \otimes_{\mathbb{F}} \mathbb{F}[[x_1, \dots, x_n]]$. In what follows, we will be dealing with formal power series from spaces of the form $V[[x, x^{-1}]]$, where V is some vector space. In the special case when $V = \mathbb{F}$, the usual multiplication of two power series of this type is not always well-defined and so one must be careful when multiplying two such series.

We now proceed directly to the definition of vertex algebra. Though they may seem unnatural at first, these properties describe a great deal of infinite dimensional algebraic objects. A *vertex algebra* consists of a vector space V over $\mathbb{F}$ with the following axioms:

1. (Vertex operator map) There exists a linear map

$$Y(\cdot, x) : V \to (\text{End } V)[[x, x^{-1}]]$$

$$v \mapsto \sum_{n \in \mathbb{Z}} v_n x^{-n-1}$$

such that for all $u, v \in V$, $v_n u = 0$ for n sufficiently large. Equivalently, there exists a linear map $V \otimes V \to V((x))$ where the codomain is the set of formal Laurent series with coefficients in V.

2. (Vacuum vector) There is a distinguished vector $\mathbf{1} \in V$ so that $Y(\mathbf{1}, x) = id_V$ and for any $v \in V$,

$$Y(v, x)\mathbf{1} = v + \sum_{n \geq 1} v_{-n-1}\mathbf{1}x^n.$$

3. (Jacobi identity) For any $u, v \in V$ and fixed $r, s, t \in \mathbb{Z}$,

$$\sum_{n \geq 0} (-1)^n \binom{r}{n} (u_{s+r-n} v_{t+n} - (-1)^r v_{t+r-n} u_{s+n}) = \sum_{n \geq 0} \binom{s}{n} (u_{r+n} v)_{s+t-n}$$

as operators on V.(We also record here an alternate form of the Jacobi identity that we will explain a little later:

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) - x_0^{-1} \delta \left(\frac{-x_2 + x_1}{x_0} \right) Y(v, x_2) Y(u, x_1)$$

$$= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2)$$

where x_0, x_1, and x_2 are formal commuting variables.)

We denote the vertex algebra by $(V, Y, \mathbf{1})$. Note that both sides on the identity in (3) are well-defined as operators on V. Later on, we will reformulate the 3rd axiom and justify calling it the Jacobi identity for vertex algebras. Finally, it is worthwhile to notice that axiom (2) implies that the vertex operator map Y is injective.

To define vertex operator algebras, it is helpful to first recall the definition of the Witt algebra and the Virasoro algebra. The *Witt algebra* $\mathfrak{d}$ is the Lie algebra with basis $d_n, n \in \mathbb{Z}$ and commutation relations $[d_m, d_n] = (m-n)d_{m+n}$ for $m, n \in \mathbb{Z}$. It can be realized as the Lie algebra of polynomial derivations $d_n = -t^{n+1}\frac{d}{dt}$, $n \in \mathbb{Z}$, on the space of Laurent polynomials $\mathbb{C}[t, t^{-1}]$. The Witt algebra is the Lie algebra of polynomial vector fields on S^1. The Witt algebra has a central extension $\mathfrak{v}$ called the *Virasoro algebra* which has basis $\{c, L_n : n \in \mathbb{Z}\}$, and commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c$$

$$[c, \mathfrak{v}] = 0.$$

This central extension is universal in the sense that for every central extension $\mathfrak{d}'$ of $\mathfrak{d}$, the natural quotient map $\mathfrak{v} \to \mathfrak{d}$ factors uniquely through $\mathfrak{d}'$.

A *vertex operator algebra* is a vertex algebra $(V, Y, \mathbf{1})$ over $\mathbb{F}$ satisfying:

1. (Conformal vector) There exists a vector $\omega \in V$, called the *conformal vector*, with

$$Y(\omega, x) = \sum_{n\in\mathbb{Z}} \omega_n x^{-n-1} = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$$

where the operators $L(n)$ satisfy the Virasoro commutation relations

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}c_V\delta_{m+n,0},$$

and $c_V \in \mathbb{F}$ is a constant called the *central charge* of V.

2. ($L(0)$ grading) The operator $L(0)$ acts diagonally on V with integer eigenvalues, i.e.,

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)},$$

where $V_{(n)} = \{v \in V : L(0)v = nv\}$, the $V_{(n)}$ are finite dimensional, and $V_{(n)} = 0$ for $n << 0$. The eigenvalues of $L(0)$ are called *conformal weights*.

3. ($L(-1)$-derivative property) For every $v \in V$,

$$Y(L(-1)v, x) = \frac{d}{dx}Y(v, x).$$

One of the more challenging things to verify is the Jacobi identity and so we will discuss two different formulations that appear commonly in the literature. The Jacobi identity as it appears above was first formulated by R. Borcherds. The formal calculus approach pioneered by I. Frenkel, J. Lepowsky, A. Meurman, H. Li, and others gives a very concise and elegant presentation of the Jacobi identity. Another approach involves the useful concept of *locality* of vertex operators. Before discussing them, we will introduce some purely algebraic techniques that will come in handy.

1.3 Formal Calculus Identities

We will develop a calculus of formal infinite series that plays a hugely important role in vertex algebra theory and is of independent interest in its own right. Recall that

$$\mathbb{F}[[x]] = \left\{ \sum_{n\geq 0} a_n x^n | a_n \in \mathbb{F} \right\}$$

is the *power series ring* with coefficients in $\mathbb{F}$. We extend this concept by considering the vector space (no longer a ring!) of *doubly infinite Laurent series* with coefficients in $\mathbb{F}$

$$\mathbb{F}[[x, x^{-1}]] := \left\{ \sum_{n\in\mathbb{Z}} a_n x^n | a_n \in \mathbb{F} \right\}$$

and slightly generalize by allowing the coefficients to lie in a vector space or algebra V

$$V[[x, x^{-1}]] := V \otimes_{\mathbb{F}} \mathbb{F}[[x, x^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} v_n x^n | v_n \in V \right\}.$$

When V contains a copy of the field $\mathbb{F}$, then we can define the *formal delta series* $\delta(x) := \sum_{n\in\mathbb{Z}} x^n$.

We adopt the so-called *binomial expansion convention*

$$(a + b)^n = \sum_{m\geq 0} \binom{n}{m} a^{n-m} b^m.$$

Note that when using the binomial expansion convention $(a + b)^n \neq (b + a)^n$ whenever $n < 0$ since the exponents of the *second* listed variable are always non-negative. Technically, we should regard $(a+b)^n$ as a function $B_n(a, b)$, but the latter notation is a bit cumbersome. This allows us to write the following useful identity.

Proposition 1 (Formal Taylor Theorem) *Let y be an (commuting) indeterminate, V a vector space, and $f \in V[[x, x^{-1}]]$ then*

$$e^{y\frac{\partial}{\partial x}} f(x) = f(x + y).$$

Proof. To prove this, first note that the both sides of the identity exist and so it suffices to prove it for $f(x) = x^n$. Note that

$$e^{y\frac{\partial}{\partial x}} x^n = \sum_{m \geq 0} \binom{n}{m} x^{n-m} y^m$$

which, according to the binomial expansion convention, can be expressed as $(x+y)^n$, proving the proposition. ■

Recall that a derivation d on a algebra A is a linear map such that $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. The formal Taylor expansion almost immediately implies the so-called "automorphism property" of e^{yT} where T is a derivation of $\mathbb{F}[x, x^{-1}]$. (Show that every derivation of $\mathbb{F}[x, x^{-1}]$ is of the form $p(x)\frac{d}{dx}$ where $p(x) \in \mathbb{F}[x, x^{-1}]$.)

Corollary 2 *Let V be an algebra, $f, g \in V[[x, x^{-1}]]$, T a derivation of $\mathbb{F}[x, x^{-1}]$, and y an indeterminate. If the product $f(x)g(x)$ exists, then the product $\left(e^{yT} f(x)\right)\left(e^{yT} g(x)\right)$ exists and*

$$e^{yT} f(x)g(x) = \left(e^{yT} f(x)\right)\left(e^{yT} g(x)\right).$$

More surprisingly, this implies the following property.

Corollary 3 *Let V be an algebra, $f, g \in \mathbb{F}[[x, x^{-1}]]$, T a derivation of $\mathbb{F}[x, x^{-1}]$, and y an indeterminate. If the composition $(f \circ g)(x) = f(g(x))$ exists, then*

$$e^{yT} f(g(x)) = f\left(e^{yT} g(x)\right).$$

This corollary can be proved by noting that it is clearly true when $f(x) = x^n$, since then

$$e^{yT} f(g(x)) = e^{yT} (g(x))^n = \left(e^{yT} g(x)\right)^n.$$

The reader is encouraged to finish the proof.

Note that we can write

$$(*) \qquad x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) = \sum_{n \in \mathbb{Z}} (x_1 - x_2)^n x_0^{-n-1}$$

$$= \sum_{n \in \mathbb{Z}, m \in \mathbb{N}} (-1)^m \binom{n}{m} x_0^{-n-1} x_1^n x_2^m.$$

The following identities feature prominently in the theory.

Proposition 4

$$x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)$$

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right).$$

They can be proved by using the following identity $(-1)^m\binom{n}{m} = \binom{-n+m-1}{m}$ and the expansion $(*)$. Alternately, they can be proved by noting that for $n \geq 1$,

$$\frac{1}{n!}\left(\frac{\partial}{\partial x_2}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = (x_1-x_2)^{-n-1} - (-x_2+x_1)^{-n-1} = \frac{(-1)^n}{n!}\left(\frac{\partial}{\partial x_1}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)$$

and that the generating function of these expressions in x_0 coincide with the different parts of the listed identities. In fact, the generating functions of the far ends can be written

$$e^{x_0\frac{\partial}{\partial x_2}} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = e^{-x_0\frac{\partial}{\partial x_1}} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)$$

which follows from the fact that

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = 0.$$

Using the ad-hoc notation $\delta_{x_0,x_1,x_2} := x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)$, then the above identities can be written

$$\delta_{x_1,x_0,-x_2} = \delta_{x_0,x_1,x_2}$$

$$\delta_{x_0,x_1,x_2} + \delta_{-x_0,x_2,x_1} = \delta_{x_2,x_1,x_0}$$

which shows a type of skew symmetry in the δ function on 3 variables.

1.4 The Jacobi Identity

Now that we have the language of formal calculus, we can write the Jacobi identity as

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) Y(u,x_1)Y(v,x_2) - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y(v,x_2)Y(u,x_1)$$

$$= x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) Y(Y(u,x_0)v,x_2)$$

where x_0, x_1, and x_2 are formal commuting variables. Verify for yourself that the coefficient of $x_0^{-r-1}x_1^{-s-1}x_2^{-t-1}$ is the expression given in axiom 3 of vertex algebras. The advantage conferred by expressing it this way is that techniques of formal calculus allow for more 'global' manipulations of the Jacobi identity. As promised, the above form of the Jacobi identity not only looks like the Jacobi identity for Lie algebras when wrriten as

$$ad\ x\ ad\ y\ (z) - ad\ y\ ad\ x(z) = ad\ (ad\ x\ y)(z)$$

where $ad\ x(y) = [x,y]$, but the coefficient of $x_0^{-1}x_1^{-m-1}x_2^{-n-1}$ is

$$u_m v_n - v_n u_m = \sum_{k\geq 0}\binom{m}{k}(u_k v)_{m+n-k},$$

and when $m = 0$, $u_0 v_n - v_n u_0 = (u_0 v)_n$. It follows that the operators v_0, for all $v \in V$, V a vertex algebra, form a Lie algebra of endomorphisms acting on V. Later we will see examples of vertex operator algebras derived from Lie algebras $\mathfrak{g}$ where these operators v_0 form a copy of $\mathfrak{g}$.

Using the Jacobi identity, we can derive two important properties of vertex algebras: (generalized) commutativity and associativity.

Definition 5 *Let V be a vertex algebra and $u, v \in V$, then u and v are* weakly commutative *if there is a $k \in \mathbb{N}$, such that*

$$(x_1 - x_2)^k [Y(u, x_1), Y(v, x_2)] = 0.$$

Proposition 6 *Let V be a vertex algebra, then the Jacobi identity implies weak commutativity for all pairs $u, v \in V$.*

Proof. Using the Jacobi identity after multiplying by a yet to be specified non-negative power of x_0 we get

$$x_0^{k-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2) - x_0^{k-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y(v, x_2)Y(u, x_1)$$

$$= (x_1 - x_2)^k \left(x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y(v, x_2)Y(u, x_1)\right)$$

$$= x_0^k x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2).$$

Note that we can replace x_0^k by $(x_1 - x_2)^k$ on the LHS since $k \geq 0$ and choose k large enough so that $x_0^k Y(u, x_0)v$ has only non-negative powers of x_0. The result follows by taking the coefficient of x_0^{-1}. ∎

Note that k depends only on u and v. For the corresponding notion for associativity, we would like to say something like "$x_1^l Y(Y(u, x_1 - x_2)v, x_2) = x_1^l Y(u, x_1)Y(v, x_2)$" but the LHS is not well-defined. Instead, we have the following definition.

Definition 7 *Let V be a vertex algebra and $u, v \in V$, then u and v are* weakly associative *if for every $w \in V$, there is an $l \in \mathbb{N}$, such that*

$$(x_0 + x_2)^l Y(Y(u, x_0)v, x_2)w = (x_0 + x_2)^l Y(u, x_0 + x_2)Y(v, x_2)w.$$

Proposition 8 *Let V be a vertex algebra, then the Jacobi identity implies weak associativity for all pairs $u, v \in V$.*

Proof. First, multiply both sides by a power l of x_1 such that $x_1^l Y(u, x_1)w$ has no pole, then note that

$$x_1^l x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2)w - x_1^l x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y(v, x_2)Y(u, x_1)w$$

$$= (x_0+x_2)^l x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_0+x_2)Y(v,x_2)w - x_1^l x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y(v,x_2)Y(u,x_1)w$$

$$= x_1^l x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2)w$$

$$= (x_0+x_2)^l x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2)w.$$

Taking the coefficient of x_1^{-1} of the second and fourth lines gives the identity.
■

Note that l depends on both u and w but not on v.

In fact, we can say much more about weak commutativity and weak associativity. We will show that the right and left hand sides of both identities are Laurent expansions of the same Laurent series with respect to different variables. To make this precise, we will use the following notation

$$\lim_{x\to a+b}\sum_{n\in\mathbb{Z}}\gamma_n x^{-n-1} = \sum_{n\in\mathbb{Z}}\gamma_n (a+b)^{-n-1}$$

whenever the right hand side is well-defined. In the usual case, we will use $\lim_{x_i\to x_j\pm x_k} f(x_0,x_1,x_2)$ for $f\in\mathbb{C}[[x_0^{\pm1},x_1^{\pm1},x_2^{\pm1}]]$.

Proposition 9 *Let V be a vertex algebra, $u,v,w\in V$, and $w'\in V^*$, then there exists a Laurent series*

$$f(x_0,x_1,x_0)\in\mathbb{C}((x_0,x_1,x_2))\subseteq\mathbb{C}[[x_0^{\pm1},x_1^{\pm1},x_2^{\pm1}]]$$

such that

$$\langle w',Y(u,x_1)Y(v,x_2)w\rangle = \lim_{x_0\to x_1-x_2} f(x_0,x_1,x_2)$$

$$\langle w',Y(v,x_2)Y(u,x_1)w\rangle = \lim_{x_0\to -x_2+x_1} f(x_0,x_1,x_2)$$

$$\langle w',Y(u,x_0+x_2)Y(v,x_2)w\rangle = \lim_{x_1\to x_0+x_2} f(x_0,x_1,x_2)$$

$$\langle w',Y(Y(u,x_0)v,x_2)w\rangle = \lim_{x_1\to x_2+x_0} f(x_0,x_1,x_2).$$

Alternately, if V is a vector space satisfying all of the axioms of a vertex algebra except for the Jacobi identity and if it satisfies both weak commutativity and weak associativity, then the above identities also hold.

Proof. By the proof for weak commutativity, we know that

$$(x_1-x_2)^k\langle w',Y(u,x_1),Y(v,x_2)w\rangle = (x_1-x_2)^k\langle w',Y(v,x_2),Y(u,x_1)w\rangle$$

for some $k\in\mathbb{N}$. Since the left side has only finitely many negative powers of x_2 and the right side has only finitely many negative powers of x_1, it follows that they are equal to a Laurent series $h(x_1, x_2)$. It is straightforward to check that the following products are well-defined

$$(x_1 - x_2)^{-k} \left((x_1 - x_2)^k \langle w', Y(u, x_1), Y(v, x_2)w\rangle\right) = \langle w', Y(u, x_1), Y(v, x_2)w\rangle$$

$$(-x_2 + x_1)^{-k} \left((x_1 - x_2)^k \langle w', Y(u, x_1), Y(v, x_2)w\rangle\right) = \langle w', Y(u, x_1), Y(v, x_2)w\rangle,$$

and so it follows that $f(x_0, x_1, x_2) = x_0^{-k} h(x_1, x_2)$ satisfies the first two identities. The third one is a consequence of the fact that the limit $\lim_{x_1 \to x_0 + x_2}$ is well defined on the first identity. Finally, weak associativity tells us that there is some $\ell \in \mathbb{N}$ such that

$$(x_0 + x_2)^\ell \langle w', Y(Y(u, x_0)v, x_2)w\rangle = (x_0 + x_2)^\ell \langle w', Y(u, x_0 + x_2)Y(v, x_2)w\rangle.$$

Now suppose that

$$f(x_0, x_1, x_2) = \frac{F(x_0, x_1, x_2)}{x_0^k x_1^l x_2^m}$$

where F has only non-negative powers of the x_i's, then we have two cases. If $\ell \le l$, then we can multiply both sides by $(x_0 + x_2)$ until we get the same equality except with l instead of ℓ. If $\ell \ge l$, then using the first identity we have that

$$(x_0 + x_2)^\ell \langle w', Y(Y(u, x_0)v, x_2)w\rangle =$$

$$((x_0 + x_2) - x_2)^{-k} (x_0 + x_2)^{\ell - l} x_2^{-m} F((x_0 + x_2) - x_2, x_0 + x_2, x_2).$$

It is a straightforward exericse to show that $((x_0 + x_2) - x_2)^n = x_0^n$ for any $n \in \mathbb{Z}$, whence it follows that

$$\langle w', Y(Y(u, x_0)v, x_2)w\rangle = x_0^{-k}(x_2 + x_0)^{-l} x_2^{-m} F(x_0, x_0 + x_2, x_2)$$

$$= \lim_{x_1 \to x_2 + x_0} f(x_0, x_1, x_2),$$

as desired.

The last statement is true because the above proof only used weak commutativity and weak associativity. ■

When V is a graded vertex algebra, like a vertes operator algebra, when we choose $w' \in \bigoplus_{n \in \mathbb{Z}} \left(V_{(n)}\right)^* \subseteq V^*$ in the setting of the above result, the Laurent series is in fact a Laurent polynomial. In this case, the result above is referred to as *rationality*.

Finally, we can show that the Jacobi identity is a consequence of formal commutativity and associativity.

Theorem 10 *Suppose that we have a space V satisfying all of the axioms of a vertex algebra except for the Jacobi identity. If (V, Y) satisfies both weak commutativity and weak associativity, then the Jacobi identity also holds.*

Proof. We will be using the notation from proposition 9. Notice that

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)$$

and so, after multiplying both sides by the series $f(x_0, x_1, x_2)$ and applying the identities

$$x_i^{-1}\delta\left(\frac{x_j-x_k}{x_i}\right) f(x_i, x_j, x_k) = \lim_{x_i \to x_j - x_k} f(x_i, x_j, x_k)$$

for the appropriate i, j, k, we get the identity

$$\left\langle w', x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) Y(u,x_1)Y(v,x_2)w - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y(v,x_2)Y(u,x_1)w \right\rangle$$

$$= \left\langle w', x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) Y(Y(u,x_0)v,x_2)w \right\rangle.$$

Since w', u, v, w are abitrary, this proves the Jacobi identity. ■

To be continued...