CLOSED SUBGROUP THEOREM FOR LIE GROUPS

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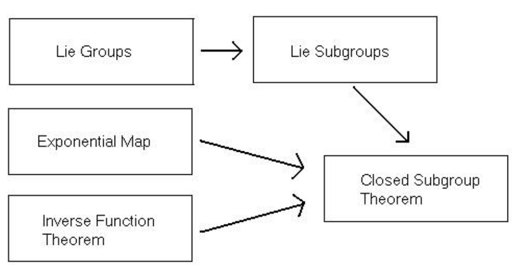
1. Introduction

Lie groups are defined on three fundamental levels: algebraic, topological, and differentiable. For G to be a Lie group, G must be a group, a topological space, and a smooth manifold. Given a subgroup H, in the purely algebraic sense, what conditions must H satisfy for it to be a Lie subgroup? As an example of how fantastically a subgroup can fail to be a Lie subgroup, consider the following example.

Example 1.1. The Lie group $G = \mathbb{R}$ (under addition) is a $\mathbb{Q}$ vector space with uncountable basis $\mathcal{B} = \{e_\lambda\}_{\lambda \in \Lambda}$. Let $e_{\lambda_0} \in \mathbb{Q}$ be a basis element and consider the subgroup H generated by $\mathcal{B} \setminus \{e_{\lambda_0}\}$. Note that $\mathbb{Q} \cap H = \emptyset$. Suppose that H is a Lie group in its own right with smooth inclusion map i, then a non-constant curve $\gamma : \mathbb{R} \to H$ in H gives a non-constant curve $i \circ \gamma$ in G. But this is impossible since the rational numbers are dense in $\mathbb{R}$! Since γ must be constant, H must have the discrete topology. Manifolds are second countable while H is uncountable, so H does not admit a Lie group structure.

Since subgroups can fail to admit a Lie group structure without topological constraints, can subgroups with *favorable* topological conditions fail to be smooth? It seems perfectly plausible for a subsets of G with only algebraic and topological conditions to fail to have compatible differentiable properties. Nonetheless, somewhat miraculously, it turns out that being a *closed* subgroup is a sufficient condition for the subgroup to have a smooth structure that makes it into a Lie subgroup. Below is a general plan for our investigation of this phenomenon.



2. Preliminaries

Here are some results that are needed for the closed subgroup theorem. We prove only the, possibly, less well-known one. We assume familiarity with the definitions of Lie group and Lie subgroup (and with basic topology and differential geometry). Throughout this text, let G be a Lie group, T_eG be its tangent space at identity, and $\exp : T_eG \to G$ be the exponential map. We consider a **Lie subgroup** H of a Lie group G to be a subgroup (algebraically) which admits a Lie group structure that makes the inclusion map $H \hookrightarrow G$ a smooth immersion. We differentiate the case when a Lie subgroup is an embedded submanifold (i.e., when the inclusion map is also a topological embedding) by calling it an embedded Lie subgroup.

Proposition 2.1. *If a Lie subgroup $H \subseteq G$ closed, then it is an embedded Lie subgroup.*

Proposition 2.2. *The exponential map is smooth and maps a neighborhood of $0 \in T_eG$ diffeomorphically onto a neighborhood of $e \in G$.*

Proposition 2.3. *The differential $d(exp)_0 : T_0(T_eG) \to T_eG$ at $0 \in T_e(T_eG)$ is the identity map (under the canonical identification of $T_0(T_eG)$ and T_eG).*

Proposition 2.4. *If $u, v \in T_eG$, then there is a smooth function $Z : \mathbb{R} \to T_eG$ so that*

$$\exp(tu)\exp(tv) = \exp(t(u+v) + t^2 Z(t)).$$

Proof. Consider the smooth map $\varphi : \mathbb{R} \to T_eG$ given by

$$\varphi(t) = \exp^{-1}\left(\exp(tu)\exp(tv)\right).$$

Since it is the following composition

$$\mathbb{R} \xrightarrow{(\exp(tu), \exp(tv))} G \times G \xrightarrow{\mu} G \xrightarrow{\exp^{-1}} T_eG,$$

where μ is the multiplication map of the group, the derivative is $\varphi'(0) = u + v$. It follows by Taylor's theorem that $\varphi(t) = t(u+v) + t^2 Z(t)$ for some smooth map Z. □

Theorem 2.5 (Inverse Function Theorem for Smooth Manifolds)**.** *If M, N are smooth manifolds and $f : M \to N$ is a smooth map, then at every point $p \in M$ where df_p is invertible there are neighborhoods $U_p \subseteq M$, $V_{f(p)} \subseteq N$ of $p, f(p)$ respectively so that the restriction map*

$$f|_{U_p} : U_p \to V_{f(p)}$$

is a diffeomorphism.

3. The Closed Subgroup Theorem

Theorem 3.1. *Let G be a Lie group and H a closed subgroup of G, then H is a Lie subgroup of G.*

Remark 3.2. The proof has four main steps which we enumerate below. The first is a lemma that exploits the density of the rational numbers and, in a certain sense, shows how this helps in the marriage of analysis and algebra. The second shows how exp is used to construct the tangent space to H, even before knowing that H is a submanifold! The last couple of steps uses exp to find a neighborhood of identity in H which will be translated around H to form a compatible atlas.

Proof. 1. *Lemma*: If $\{u_n\}_n$ is a sequence in T_eG such that $\frac{u_n}{||u_n||} \to v \in T_eG$, $||u_n|| \to 0$, and $\exp u_n \in H$ for all n, then $\exp tv \in H$ for all $t \in \mathbb{R}$.

Proof of lemma: Clearly, $\frac{tu_n}{||u_n||} \to tv$. Since $||u_n|| \to 0$ we can find integers m_n so that $m_n||u_n|| \to t$ (we can do this by finding sequences in $\mathbb{Q}$ so that $a_n/b_n \to t$ and $|||u_n|| - p_n/q_n| \to 0$, then setting $m_n =$ the integer part of $\frac{a_n q_n}{b_n p_n}$). By the continuity of exp and the fact that U is a closed subgroup, we have that

$$H \ni (\exp(u_n))^{m_n} = \exp\left(m_n||u_n|| \cdot \frac{u_n}{||u_n||}\right) \to \exp(tv) \in H.$$

2. The set V of all v such that $\exp(tv) \in H$ can be shown to be a subspace of T_eG. Suppose that $v_1, v_2 \in V$ where $v_1 \neq -v_2$. We would like to show that $t(v_1+v_2) \in V$ for all $t \in \mathbb{R}$. To see this, fix t, let n be a large positive integer, and recall that we can find a smooth function Z so that

$$\left(\exp\left(\frac{t}{n}v_1\right)\exp\left(\frac{t}{n}v_2\right)\right)^n = \left(\exp\left(\frac{t}{n}(v_1+v_2) + \frac{t^2}{n^2}Z\left(\frac{t}{n}\right)\right)\right)^n$$

$$= \exp\left(t(v_1+v_2) + \frac{t^2}{n}Z\left(\frac{t}{n}\right)\right) \in H$$

and so taking $n \to \infty$ and exploiting the closedness of H shows that $\exp(t(v_1+v_2)) \in H$. It follows that V is a vector subspace of T_eG.

3. We show that the set $\exp V$ is a neighborhood of e in H. Let $U \subsetneq T_eG$ be a complementary subspace of V and consider the map $\gamma(u,v) = \exp(u)\exp(v)$ where $u \in U$, $v, \in V$. It is smooth since it is the composition of smooth maps. It is also a diffeomorphism on a neighborhood of identity (check the differential and use the inverse function theorem). We would like to show that $\exp(V) = \gamma(V)$ is a neighborhood of identity in H. Suppose otherwise, then we can find a sequence (u_n, v_n) so that $\gamma(u_n, v_n) \in H$ and $\gamma(u_n, v_n) \to e$ with $u_n \neq 0$ for all n. Since H is a subgroup, it follows that $\exp(u_n) \in H$ for all n. Passing to a subsequence $\{u_{n_k}\}_k$ we can find a $u \in U$ with $|u| = 1$ so that $\frac{u_{n_k}}{|u_{n_k}|} \to u$ (by the compactness of the unit ball in $\mathbb{R}^N$). But then, by construction, $u \in V$, contradicting the fact that $U \cap V = \{0\}$. This proves that $\exp(V)$ is indeed a neighborhood of e in H.

4. To show that H is an embedded submanifold, we need only find an atlas of H so that the inclusion map is a smooth embedding. Letting B be an open ball about $0 \in V$ so that $\exp B$ is diffeomorphic to an open neighborhood of $e \in H$,

we can use left translation to give every point in H an open neighborhood that is diffeomorphic to B. The map $B \to H \hookrightarrow G$ is a smooth embedding since exp is smooth, so the result follows immediately. □

4. References

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