The Brouwer Fixed Point Theorem

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1 Introduction

We explore some proofs of the famous fixed point theorem of Brouwer. A cursory search will yield various proofs from various fields, each interesting in its own right. The proliferation of proofs is a testament to the theorem's widespread impact and fundamental nature. The first proof presented here, due to M. Hirsch, is geometric and so has a more accessible nature than the others. The second involves algebraic topology, namely the use of homology, and follows immediately from computing the homology groups for well-known spaces. The last one is a rather surprising combinatorial proof due to E. Sperner.

2 Differential Geometry

To facilitate the proof of the theorem from a differential geometry viewpoint, we recall some basic results from analysis and the theory of real smooth manifolds. The first is the Stone-Weierstrass approximation theorem, but we will be more interested in the corollary below it. The second is the ever-useful Sard's theorem. The third is an intuitive result on level sets in manifolds with boundary. Finally, the fourth is the classification of smooth compact 1-dimensional manifolds. We state these without proof, but some references are included.

Theorem 2.1 (Stone-Weierstrass Approximation Theorem)**.** *Let* X *be a compact metric space and* $A \subseteq C^0(X, \mathbb{R})$ *a unital subalgebra such that for every distinct pair of points* $x, y \in X$ *there is an element* $f \in A$ *with* $f(x) \neq f(y)$*, then* A *is dense in* $C^0(X, \mathbb{R})$*.*

Corollary 2.2. *Let $X \subseteq \mathbb{R}^n$ be compact, then the set of polynomials in n variables $P(X, \mathbb{R})$ is dense in $C^0(X, \mathbb{R})$.*

Theorem 2.3 (Sard's Theorem)**.** *Let M, N be smooth manifolds (with or without boundary) and $f : M \to N$ a smooth map, then the set of critical values of f has measure zero in N.*

Proof. See [Lee13]. $\square$

The specialized result that follows implies as a corollary the classification of compact one-dimensional smooth manifolds. If M is such a manifold, it suffices to restrict our attention to one of its finitely many connected components.

Theorem 2.4. *Let M be a compact, connected one-dimensional smooth manifold (with or without boundary), then M is diffeomorphic to S^1 if $\partial M = \emptyset$ and is diffeomorphic to $[0, 1]$ otherwise.*

Proof. See [Aud14] for interesting proof. $\square$

In general, it can be shown that connected one-dimensional smooth manifolds are diffeomorphic to either S^1 or one of the intervals $(0, 1), [0, 1), (0, 1], [0, 1]$ [Mil65]. Finally, we record a lemma before proving Brouwer's fixed point theorem.

Lemma 2.5. *Let M be a smooth manifold with boundary, N a smooth manifold, and $f : M \to N$ a smooth map. If $y \in N$ is a regular value of both f and $f|_{\partial M}$, then $\partial f^{-1}(y) = f^{-1}(y) \cap \partial M$.*

Theorem 2.6 (Brouwer's fixed point theorem)**.** *Let D^n be the closed ball in $\mathbb{R}^n$ and $\varphi : D^n \to D^n$ be a continuous map, then φ has a fixed point.*

Proof. Suppose that φ does not have a fixed point and let $\tilde{\varphi} : D^n \to D^n$ be a smooth map that approximates φ and also lacks a fixed point. To show that we can find such a smooth map, we invoke the Stone-Weierstrass approximation theorem above, which tells us that we can find a polynomial map $\alpha : D^n \to \mathbb{R}^n$ such that $||\alpha(x) - \varphi(x)|| < \varepsilon$ for any $\varepsilon > 0$. It is possible that the image of α does not lie in D^n, and so as a correction we define

$$\tilde{\varphi} : D^n \to D^n, \quad x \mapsto \frac{\alpha(x)}{(1+\varepsilon)}.$$

Notice that

$$||\alpha(x) - (1+\varepsilon)\varphi(x)|| < 2\varepsilon$$

and so $||\tilde{\varphi} - \varphi|| < 2\varepsilon$. Finally, to ensure that $\tilde{\varphi}$ has no fixed points, choose ε so that $\varepsilon < \frac{1}{2}\min_{x \in D^n} ||\varphi(x) - x||$.

Using $\tilde{\varphi}$ we construct a retraction $\rho : D^n \to S^{n-1}$ onto the boundary of D^n by sending $P \in D^n$ to the point $\vec{QP} \cap S^{n-1}$ where $\tilde{\varphi}(P) = Q$ and $\vec{QP}$ is the ray starting at Q which goes through P. To see that ρ is smooth explicitly, let

$$v = \frac{x - \tilde{\varphi}(x)}{||x - \tilde{\varphi}(x)||}$$

then the quadratic equation gives

$$\rho(x) = \tilde{\varphi}(x) + tv \text{ where } t = -\tilde{\varphi}(x) \cdot v + \sqrt{(\tilde{\varphi}(x) \cdot v)^2 - \tilde{\varphi}(x) \cdot \tilde{\varphi}(x) + 1}.$$

Now, by Sard's theorem, we can find a regular value $a \in S^{n-1}$ of ρ. Since ρ restricts to identity on S^{n-1}, a is a regular value for $\rho|_{S^{n-1}}$ and it follows from lemma 2.5 that the fiber $\rho^{-1}(a)$ is a compact 1-dimensional manifold with boundary $\partial\rho^{-1}(a) = \rho^{-1}(a) \cap \partial D^n = \{a\}$. But every compact 1-dimensional manifold is the disjoint union of finitely many copies of S^1 and finitely many copies of $[0, 1]$. In other words, the number of boundary points of $\rho^{-1}(a)$ must be even, a contradiction. $\square$

3 Algebraic Topology

Before we prove the theorem from a topological point of view, we need to recall some important constructions and results. First and foremost is that of the *homology groups of topological spaces*. In a sense, the substantial machinery of homology simplifies the proof a great deal while somewhat bypassing geometric intuition. For proofs of the results below, see [Hat02].

Proposition 3.1. *Let S^n be the n-dimensional sphere, then*

$$H_k(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, n \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 3.2. *Let X be a contractible topological space, then $H_k(X) = 0$ for $k \geq 1$ and $H_0(X) \cong \mathbb{Z}$.*

Theorem 3.3 (Brouwer's fixed point theorem)**.** *Let D^n be the closed ball in $\mathbb{R}^n$ and $\varphi : D^n \to D^n$ be a continuous map, then φ has a fixed point.*

Proof. Let $\varphi : D^n \to D^n$ be a continuous map and assume that φ has no fixed points. Construct a retraction $\rho : D^n \to S^{n-1}$ as in theorem 2.6. If $n = 1$, then the image of φ is not connected, a contradiction. Assume $n > 1$, then the identity map $id_{S^{n-1}}$ factors through D^n $S^{n-1} \hookrightarrow D^n \xrightarrow{\rho} S^{n-1}$ and the identity map $\mathbb{Z} \to \mathbb{Z}$ factors similarly

$$\mathbb{Z} \cong H_{n-1}(S^{n-1}) \to H_{n-1}(D^n) \xrightarrow{\rho_*} H_{n-1}(S^{n-1}) \cong \mathbb{Z}.$$

But D^n is contractible and so the middle term is zero by proposition 3.2, a contradiction. Therefore, φ must have a fixed point. ☐

4 Combinatorial Approach

It is always refreshing to see discrete methods used in topological problems to great effect. To apply Sperner's ideas to the fixed point theorem, we need one main tool: Sperner's lemma.

Definition 4.1. *An* <u>n-dimensional simplex</u> *(or simply* <u>n-simplex</u>*) is the convex hull of $n+1$ points in general position. We give n-simplices coordinates of the form $(x_1, ..., x_{n+1})$ such that $0 \leq x_i$ for all i and $\sum_{i=1}^{n+1} x_i = 1$. An* <u>m-face</u> *of an n-simplex S is the convex hull of a size $m+1$ subset of the $n+1$ points defining S.*

Definition 4.2. *A* simplicial subdivision *S' of an n-simplex S is a partition of S into n-simplices called* cells *such that, taken pairwise, cells are either disjoint or have intersection equal to a face (of some dimension).*

Definition 4.3. *A* proper coloring *of a simplicial subdivision S' of an n-simplex S is a coloring of the vertices of S' with $n+1$ colors such that every color is used and whenever a vertex v lies on a face F of S where $F =$ convex hull$(x_1, ..., x_k)$, the color of v is the same as the color of x_i for some $i (1 \leq i \leq k)$.*

Lemma 4.4 (Sperner's Lemma)**.** *Every properly colored simplicial subdivision of an n-simplex contains a cell whose vertices all have different colors.*

Theorem 4.5 (Brouwer's fixed point theorem)**.** *Let D^n be the closed ball in $\mathbb{R}^n$ and $\varphi : D^n \to D^n$ be a continuous map, then φ has a fixed point.*

Proof. D^n is homeomorphic to an n-simplex S and so we can assume that $\varphi : S \to S$ is continuous. Suppose that it does not have a fixed point and consider a sequence of simplicial subdivisions $S_1, S_2, ..., S_k, ...$ so that the diameter of the largest cell in S_k approaches zero as $k \to \infty$. (Precisely, this can be done by taking the *barycentric subdivisions.*) Since φ has no fixed point, we can assign a coloring of each point in S by $S \ni x \mapsto i \in \{1, ..., n+1\}$ whenever $f(x)_i < x_i$ (the ith coordinate) and i is the least index for which this happens. Note that this is well defined since $\sum x_i = 1 = \sum f(x)_i$ and $f(x) \neq x$ implies that there is at least one i so that $f(x)_i < x_i$ (and at least one j so that $f(x)_j > x_j$).

To show that this is a proper coloring, first notice that the vertices $e_i = (0, 0, ..., \underbrace{1}_{i^th}, 0..., 0)$ have the ith color, satisfying the first condition that all colors are used. Second, notice that on any face $F =$ *convex hull*$(e_{i_1}, ..., e_{i_k})$ of S, if $x \in F$, then the only (potentially) nonzero coordinates are $e_{i_1}, ..., e_{i_k}$, and so x must be colored the same as one of the e_{i_j} defining F. Therefore, this gives a proper coloring.

By Sperner's lemma, for every k, S_k has a cell E_k whose vertices have all colors. Fixing the color 1, let $y_k^1 \in E_k$ be the vertex with the color 1 and, since S is compact, we can find a subsequence, which we will also denote by $\{y_k^1\}_{k=1}^{\infty}$ that converges to some element $z \in S$. Furthermore, since $diameter(E_k) \to 0$ it follows that the ith colored vertices y_j^i also converge to the same point z. By the continuity of φ, it must be the case that $\varphi(z)_i < z_i$ for all $i(1 \leq i \leq n+1)$, which is the contradiction that proves the theorem. □

5 References

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