Notes on Lecture of September 19, 2011 - in particular on filling the gap

Theorem Suppose that p is a real number and that $0 < p < 1$. There is a positive real number r such that $r^2 = p$.

Summary of the proof from lecture with a patch for the gap

We first saw an inductive definition that produced, for each positive integer n, an n-place decimal fraction x_n such that

$$0 \leq x_n = 0.d_1 d_2 ... d_n$$

where each d_k is a decimal digit, and

$$(x_n)^2 \leq p < (x_n + 10^{-n})^2.$$

Note that x_n is the largest n-place decimal fraction whose square is not larger than p. Also note that the set of all these x_n is bounded above by 1 and is non-empty. Thus the Axiom of Completeness gives us the real number r defined by

$$r = lub(\{x_n : n \in \mathbb{N}\}.$$

Since we have, for all positive integers n, $0 \leq r \leq x_n$ we can conclude that

for all positive integers n, $0 \leq (x_n)^2 \leq r^2$.

The gap I left was the absence of a proof for the claim

(C) for all positive integers $k, r^2 \leq (x_k + 10^{-k})^2$.

To prove (C) it is enough to prove

(C') for all positive integers $k, r \leq (x_k + 10^{-k})$.

For each positive integer k introduce the notation $y_k = x_k + 10^{-k}$ and note that $x_k < y_k$.

Step 1: Show that for all positive integers $n, x_n \leq x_{n+1}$ and conclude that whenever $n < m$ then $x_n \leq x_m$.

Consider an arbitrary positive integer n. Recall that $d_{n+1} \in \mathbb{D}$ and thus $0 \leq d_{n+1} + 1 \leq 10$. Now

$$x_n \leq x_n + d_{n+1} \cdot 10^{-(n+1)} = x_{n+1}.$$

Step 2: Show that for all positive integers $k, y_{k+1} \leq y_k$ and conclude that whenever $j < k$ then $y_k \leq y_j$.

$y_{k+1} = x_{k+1} + 10^{-(k+1)}$ by the definition of y_{k+1}

$y_{k+1} = [x_k + d_{k+1}\, 10^{-(k+1)}] + 10^{-(k+1)}$ by the definition of x_{k+1}

$y_{k+1} = x_k + [d_{k+1} + 1]\, 10^{-(k+1)}$ by arithmetic

$y_{k+1} \leq x_k + (9 + 1)\, 10^{-(k+1)}$ since d_{k+1} is a decimal digit

$y_{k+1} \leq x_k + 10^{-k} = y_k$ by arithmetic and the definition of y_k.

Step 3: Show that for every positive integer k, y_k is an upper bound for the set of all x_n.

Consider an arbitrary k and treat it as fixed for the moment. Now consider an arbitrary n. We work by cases.

Case 1: $n = k$. Then By step 1 we already know that $x_n = x_k < y_k$.
Case 2: $n > k$. Use the second part of Step 2. See that $x_n < y_n \leq y_k$.
Case 3: $n < k$. Use the second part of Step 1. See that $x_n \leq x_k < y_k$.

So far we know that r is the least upper bound of the set of all x_n and that each y_k is one upper bound for that set. Thus for all indices k including the case $n = k$, we have even more than (C'). We have, for all k

$$0 \leq x_k \leq r = lub(\{x_n : n \in \mathbb{N}\}) \leq y_k = x_k + d_k\, 10^{-k}$$

(C) follows from (C').

My gap is now filled. We go back to the summary of Monday's lecture. Consider an arbitrary positive integer n. We have both

$$(x_n)^2 \leq p < (x_n + 10^{-n})^2$$

and, using (C) with $k = n$,

$$(x_n)^2 \leq r^2 \leq (x_n + 10^{-n})^2.$$

Thus we get

$$0 \leq |r^2 - p| \leq (x_n + 10^{-n})^2 - (x_n)^2 = [(x_n + 10^{-n}) + (x_n)][(x_n + 10^{-n}) - (x_n)] = [2\,x_n + 10^{-n}][10^{-n}]$$

from which it follows that

$$0 \leq |r^2 - p| \leq [2\,x_n + 10^{-n}][10^{-n}] \leq (2.1)10^{-n} < 2.1 \times n^{-1}$$

since we know that $x_n \leq 1$ and $10^{-n} \leq 10^{-1}$. Divide through by 2.1.

$$0 \leq \frac{|r^2 - p|}{2.1} \leq \frac{1}{n}$$

Since n was arbitrary we have

$$0 \leq \frac{|r^2 - p|}{2.1} \leq glb(\{n^{-1} : n \in \mathbb{N}\}) = 0$$

from which we must conclude that $|r^2 - p| = 0$ and $r^2 = p$.