CORRECTION TO "C_1 IN [2] IS ZERO"

Abbas Bahri

0.Introduction.

John Morgan and G,Tian pointed out a mistake in the concluding argument for our paper entitled "C_1 in [2] is zero" [4], which was recently published in arXiv/Math/DG:1512.02098. We hereby acknowledge this mistake and correct the computation, leading to the conclusion that C_1 is non-zero and that their reference [2] does indeed fully address and resolve the counter-example which we provided in [3] to the inequality (19.10) in their monograph [1].

For the sake of completeness, we repeat here the first section of [4], providing the framework for our present computations and relating them to the framework of [4]:

1. Preliminaries.

We assume in the sequel that the curve-shortening flow, starting from a given curve, defines a piece of (immersed)surface Σ. This happens for example when $k(c(x_0,0))$, the norm of the curve-shortening flow deformation vector $H(c(x,0))$ as in eg [1], is non-zero at a given point x_0 of a smooth immersed curve $c(x,0)$. Extending in section to the curve-shortening flow, we find that an open set U in M is parameterized as $\{c_\mu(x,t)\}$, μ an extra-parameter, with $\frac{\partial c_\mu(x,t)}{\partial t} = H((c_\mu(x,t)) = \nabla_S^{g(t)} S(c_\mu(x,t))$, S is the unit vector of $x \longrightarrow c_\mu(x,t)$, $((t,\mu)$ frozen) for the metric $g(t)$ evolving as in [1] through the Ricci flow.

U is now mapped into $M \times [0,\epsilon)$ through the map $c_\mu(x,t) \longrightarrow (c_\mu(x,t),t), t \in [0,\epsilon)$.

This is the framework of [2], with the metric $\hat{g}$ on $M \times [0,\epsilon)$. The image of M through this map will be denoted M_1 in the sequel.

2. Correction to the computation of [4], page 3, line 15.

The notations, definitions etc are those of [4], with the special choices made for S $(S(c(x,t),s) = \frac{\frac{\partial c(x,t)}{\partial x}}{|\frac{\partial c(x,t)}{\partial x}|_{g(t)}}$, over $M \times [0,\epsilon))$, H ($H(c(x,t),s) = \nabla_S^{g(t)} S$, the covariant derivative along the unit vector S for $g(t)$ to the curve $x \longrightarrow (c(x,t),s)$, (t,s) frozen) etc in [4], section 2.

The mistake takes place page 3, line 15 of [4] when computing $(\hat{\nabla}_{\frac{\partial}{\partial t}} \hat{\nabla}_S S, H)$. The metric is variable here and the derivatives of the Christoffel symbols lead to a non-zero C_1. We will be completing the computation in a slightly unusual way, there is a more direct one, but we do prefer the computation which we are presenting here. We compute:

$(\hat{\nabla}_{\frac{\partial}{\partial t}} \hat{\nabla}_S S, H) = (\hat{\nabla}_{\hat{H}} \hat{\nabla}_S S, H) - (\hat{\nabla}_H \hat{\nabla}_S S, H)$

Now, $\hat{H}$ is along $(c(x,t),t)$. Thus, the metric is $g(t)$ and $\hat{\nabla}_S S = H + Ric(S,S)\frac{\partial}{\partial t}$. Thus,

$(\hat{\nabla}_{\hat{H}} \hat{\nabla}_S S, H) = (\hat{\nabla}_{\hat{H}}(H + Ric(S,S)\frac{\partial}{\partial t}), H) = (\hat{\nabla}_{\frac{\partial}{\partial t}}(H + Ric(S,S)\frac{\partial}{\partial t}), H) + (\hat{\nabla}_H(H + Ric(S,S)\frac{\partial}{\partial t}), H)$

Since, $\hat{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0$ and $(\frac{\partial}{\partial t}, H) = 0$, since $(\hat{\nabla}_H(Ric(S,S)\frac{\partial}{\partial t}), H) = O(k^2)$,

we find that:

$$(\hat{\nabla}_{\frac{\partial}{\partial t}} \hat{\nabla}_S S, H) = (\hat{\nabla}_{\frac{\partial}{\partial t}} H, H) + (\hat{\nabla}_H H, H) - (\hat{\nabla}_H \hat{\nabla}_S S, H) + O(k^2) =$$

$$= (\hat{\nabla}_H H, H) - (\hat{\nabla}_H \hat{\nabla}_S S, H) + O(k^2)$$

Now, since S is horizontal, $\hat{\nabla}_S S = \nabla_S S + \theta \frac{\partial}{\partial t}$, θ bounded, so that

$$(\hat{\nabla}_H \hat{\nabla}_S S, H) = (\hat{\nabla}_H \nabla_S S, H) + O(k^2)$$

Thus, our above expression is, up to $O(k^2)$:

$$(\nabla_H H, H) - (\nabla_H \nabla_S S, H)$$

$H(c(x,t),s)$ is equal to $\nabla_S^{g(t)} S$, with $S(c(x,t),s) = \frac{\frac{\partial c(x,t)}{\partial x}}{|\frac{\partial c(x,t)}{\partial x}|_{g(t)}}$. Along H, $(c(x,t),s)$ changes after the time τ into $(c(x,t+\tau),s)$. With $s=t$, the metric is $g(t)$, so that, along a piece of curve tangent to H as defined here:

$$\nabla_S S(c(x,t+\tau),s) = \nabla_S^{g(t)} S$$

, with $S(c(x,t+\tau),t) = \frac{\frac{\partial c(x,t+\tau)}{\partial x}}{|\frac{\partial c(x,t+\tau)}{\partial x}|_{g(t+\tau)}}$ instead of $\nabla_S S(c(x,t+\tau),s) = \nabla_S^{g(t+\tau)} S$,with $S(c(x,t+\tau),t)$ as above. This is the expression that we would find in $(\nabla_H H, H)$ and there is therefore a difference between $H(c(x,t+\tau),t)$ and $\nabla_S^{g(t)} S$, where S is taken at $(c(x,t+\tau),t)$. The difference appears through the Christoffel symbols of the two different metrics $g(t+\tau)$ and $g(t)$. In $(\nabla_H H, H) - (\nabla_H \nabla_S S, H)$, this difference is differentiated along H, that is along τ and it leaves a single factor for H, giving rise to $C_1 k$, with C_1 non-zero.

The observations of John Morgan and Gang Tian, leading to the complete resolution of this matter, are fully acknowledged here.

References

1. J.Morgan and G.Tian, *Ricci Flow and the Poincare Conjecture*, vol. 3, Clay Mathematics Monograph, AMS and Clay Institute, 2007.
2. J.Morgan and G.Tian, *Correction to Section 19.2 of Ricci Flow and the Poincare Conjecture*, arXiv:1512.00699 (2015).
3. A.Bahri, *A Counterexample to the second inequality of Corollary (19.10)in the monograph Ricci Flow and The Poincare Conjecture by J.Morgan and G.Tian* (2015).
4. A.Bahri, C_1 *in [2] is zero*, arXiv/math/DG:1512.02098 (2015).