M/v

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Abstract. Given a three dimensional closed contact manifold (M^3, α) and a nowhere singular Morse vector-field v in its kernel, we sketch the construction of the space M/v discussed in [4]. We also introduce spaces of immersed curves in M/v and an action functional on these spaces. This is the first step in the completion of a program aimed at computing the homology for contact forms defined in [2] and [7].

1. Introduction.

We consider in this paper a three dimensional closed manifold M and a contact form α on M. We assume that there is a nowhere zero vector-field v in $ker\alpha$, which we also assume to be Morse-Smale. v might have some hyperbolic orbits around which $ker\alpha$ "does not turn well", see [1], I.11, [2]. We sketch in what follows a method in order to compute the contact homology that we have defined in [2], [3]. As we have indicated in earlier papers [3], [4], this computation requires the use of the space M/v, a highly pathological, non Hausdorff space. We thus have to devote some time to define such a space, or subsets of this space in a manner that suits our purpose.

The idea here has two sides: on one hand, a proper, acceptable definition of the space of orbits mod v, M/v, cannot be given. But a "hybrid" representation of this space, using partly a section to v and partly periodic orbits can be provided.

The next step is then to consider the Z-structure over such a "section" provided by the contact structure; namely, our "section" will be defined in a v-invariant subset of M, where α "turns well" along v. Accordingly, every point x on a v-orbit originating in our "section" will have infinitely many "coincidence points", see [1], Definition 0.1, p I.7 ,[2], Definition 9, p 196 (a "coincidence point" of [2] is an "oriented coincidence point" of [1]; the discrepancy is unfortunate, but meaningless); these are points z_k such that $ker\alpha$ has rotated $k\pi$ from x to z_k, $k \in Z$ ($2k\pi$ for [2]. We will use here the original terminology of [1]).

Let ξ be the Reeb vector-field of α. Assume that $\beta = d\alpha(v, .)$ is, in this subset of M defined by the v-orbits originating at this "section", a contact form with the same orientation as α.

We define in what follows path spaces adjusted to this Z structure. These path spaces are the natural generalization of the space of immersed curves in S^2 of Maslov index zero. This space of immersed curves appears in a natural way [1], [2], when we study the standard contact structure of S^3 and we take for v a vector-field defining a Hopf fibration in its kernel.

Among all the contact forms of this contact structure, there is a special subset which corresponds to the contact forms of this contact structure which are invariant through the antipodal map. These are the "symmetric" contact forms of this contact structure. They enjoy additional symmetries and the study of their Reeb vector-fields and their periodic orbits is greatly simplified as a consequence.

The antipodal map is a special map that generalizes to the most general framework of a contact structure and of a vector-field v, maybe non-singular, of its kernel. Accordingly, the notion of a symmetric α generalizes, as we will see, albeit under some restrictions. An averaging procedure (section 2) allows to define such a notion.

We then sketch the definition in section 3 (the logical order should have been the reverse one) of the space M/v. We essentially show how to define a fundamental domain for an iteration map along v along e.g an attractive orbit of O_1 and we show how we can evolve from there and "travel" using time maps of the one parameter group of v to the hyperbolic orbits and to the repulsive ones, "filling" sections etc.

In the last section, section 4, we sketch the definition of a "symmetric" functional J_s on a space of curves slightly smaller than the space of immersed curves of M/v and we indicate why J_s should become very large or tend to ∞ as we tend to the (hyperbolic to the least, and after some adjustments [6])traces of the periodic orbits of v in M/v.

2. The path spaces, the nearly symmetric α.

Let us assume that the space M/v has been defined as an "orbifold section" to v, possibly with boundary. Typically, we would think of the standard contact structure of S^3, of v as being a Morse-Smale perturbation in its kernel of a vector-field defining a Hopf fibration, with an attractive periodic orbit O_1 and a repulsive one O_2, see [5], Theorem 1, to find such a vector-field v in an almost explicit form. M/v then can be taken to be a disk transverse to O_1, with boundary O_2.

If we then consider an immersed C^1-closed curve $x(t)$ in this "section", we can lift it above this "section" along v into a C^1-curve $y(t)$ so that $\dot{y}(t)$ reads as $a\xi + bv$, a positive and $y(t)$ is derived from $x(t)$ by v-transport. $y(t)$ is not unique, neither is it necessarily a closed curve. Rather, given one of the lifts $y(t)$, all other lifts are indexed by an integer $k \in Z$ and derived using the map along the v-orbit which assigns to a point x_0 the point x_k uniquely defined by requiring that β (thus ξ) has completed k half-revolutions between x_0 and x_k.

In order to have $y(t)$ closed, we may have to ask that $x(t)$ be a path, rather than a closed curve, starting at a point x_0 and ending at a point x_1, both in the "section" and both on the same v-orbit.

Let T be the map, defined on the "interior of the section", which assigns to x_0 of this "section" the next point x'_0 on the v-orbit through x_0 which belongs again to this "section".

If the end point x_1 of the curve $x(t)$ defined above is equal to $T^m(x_0)$ for suitable values of m and if $\dot{x}(1)$, the tangent vector to $x(t)$ at x_1, is equal to $DT^m(\dot{x}(0))$, then the curves $y(t)$ will be closed curves as we will see.

Not only the curve $x(t)$, $t \in [0,1]$, lifts into $y(t)$ and the corresponding family of closed curves above $x(t)$. The curves in this "section" defined by $T^i(x(t))$, $t \in [0,1]$, $i \in Z$ all lift into the same family of curves $y(t)$. So that we find it natural to introduce:

Definition 1. *the space of curves $\Lambda_{T^m}(M/v)$ (M/v is our "section") defined as the set of C^1-curves in M/v running from a point x_0 of M/v to the point $T^m(x_0)$.*

The map T defines a transformation T_* of this space and we will denote $\Lambda^*_{T^m}(M/v)$ the quotient of this space by T_*.

Our space could be in fact more specific because M/v is typically a stratified space of dimension 2, with boundary one of the attractive or repulsive orbits. Typically, M/v is a disk, with boundary an attractive periodic orbit O_1 for example.

We can arrange so that T^m, restricted to O_1, is the identity map and that ξ rotates exactly $m\pi$ in the v-transport along O_1. It is then natural to consider the space $(M/v)/O_1$, the topological quotient of M/v by its subset O_1 and therefore to introduce the space $\Lambda^*_{T^m}((M/v)/O_1) = \Lambda^*$ of C^1-curves running in $(M/v)/O_1$ from an initial point x_0 to $T^m(x_0)$ mod out by the map T^* (defined as above, but acting on these new spaces).

Embedded into Λ^*, we find the space of C^1-immersed curves Imm^*.

The group S^1 acts on these spaces by time translation. Furthermore, above any given curve in Imm^*, we find a family of closed curves $y(t)$, $t \in [0,1]$, with $\dot{y} = a\xi + bv$, only that the constant a might change with the curve y in the family. a does not change if the form α is "symmetric", that is if, whenever the v-transport, along a v-orbit from x_0 to x_1, maps ξ into $\lambda\xi$, then $\lambda = 1$.

Of course, given a contact structure, a vector-field v in its kernel and a contact form α in this contact structure, we cannot expect α to be "symmetric"; neither can we assume, in general, the existence of a "symmetric" α.

The nearly symmetric α.

However, after averaging α, we can assume that α is nearly "symmetric". This averaging procedure goes as follows: considering a point z_0 above M/v, we introduce the points z_i, $i \in [-N, N]$, N large. z_i is defined by the condition that it is the i^{th}-point on the v-orbit such that α is mapped onto $\lambda_i\alpha$ from z_0 to z_i. A candidate in order to replace

α at z_0 is $\frac{1}{(\overset{N}{\underset{i=-N}{\Sigma}}\frac{1}{\lambda_i})}\alpha$; this is formally an almost symmetric contact form in the same contact structure than α. It is, by Lemma 1 of [6] v-convex (that is, denoting θ this form that has v in its kernel, $d\theta(v,.)$ is also a contact form with the same orientation than θ) since each of the forms $\lambda_i\alpha$ is v-convex (they are pull-backs of α through v-transport maps); but it has the disadvantage to tend to zero as N tends to ∞ on any v-orbit that is asymptotic to an attractive or a repulsive periodic orbit of v.

The contact form $(\overset{N}{\underset{i=-N}{\Sigma}}\lambda_i)\alpha$ is also nearly symmetric and does converge on any v-orbit tending at ∞ to attractive or repulsive periodic orbits. It is, however, not necessarily v-convex.

Near an attractive or a repulsive periodic orbit of v, a model for (α, v) has been provided in [6] p47. This model can be slightly modified so that $\lambda_i = \bar{\gamma}^i$, with $0 \lneq \bar{\gamma} \lneq 1$. It follows from this model that $(\overset{N}{\underset{i=-N}{\Sigma}}\lambda_i)\alpha$ is also v-convex (terminology of [5], Lemma 1) near the attractive or repulsive periodic orbits of v (taking (α, v) according to the model).

This "nearly symmetric" form also "turns well" along v wherever $ker\alpha$ "turns well" along v. We can then use, as in [6], the second order differential equation along v:

$$(1)[v,[v,\xi]] = -\xi + \gamma(s)[\xi, v] - \gamma'(s)ds(\xi)v$$

with $s(.)$ denoting the length along v on a given piece of v-orbit with a given origin; this differential equation allows to modify α, ξ, see Lemma 1 of [6], by rescaling the rotation of $ker\alpha$ along v.

Starting from the data that we have near each attractive or repulsive periodic orbit, we can evolve along v-orbits and induce a uniform rotation, thereby deriving a nearly symmetric α outside of small tori around the attractive and repulsive periodic orbits of v, along subsets of M where sections to v can be defined; this nearly symmetric α_s is now v-convex, this is embedded in the rescaling with the use of (1); it is furthermore equal to $(\overset{N}{\underset{i=-N}{\Sigma}}\lambda_i)\alpha$ near the repulsive or attractive periodic orbits of v.

If there are hyperbolic orbits of v, global sections outside of the attractive and repulsive periodic orbits might not be available. We need therefore to remove, in a first step of in our construction, the hyperbolic orbits and their stable and unstable manifolds. The rescaling and the definition of an almost symmetric α_s can be completed on the remaining set. In order to extend the definition of our form to the hyperbolic orbits and their stable and unstable manifolds, we follow the construction of [6]; in [6], this construction was carried near a hyperbolic orbit such that $ker\alpha$ did not "turn well" along it. The general idea would be to extend it to all hyperbolic orbits . This construction needs to be carried out in great detail in the present framework (with the aim of deriving a nearly symmetric form in the vicinity of these orbits, or having the associated functional, see section 4 below, tend to ∞ as the curves come to intersect one of these hyperbolic orbits).

Similarly, although the definition of this nearly symmetric contact form is very precise near the attractive and repulsive orbits, the effect of the rescaling, completed above with the use of (1), on the associated variational problem (to be "defined" in section 4, below) and the behavior of its critical points at infinity near the repulsive or attractive periodic orbits of v need to be thoroughly understood.

3. M/v**.**

Let us now enter into more details in the construction of "M/v". We assume that v has two periodic orbits, O_1 which is a attractive and O_2 which is repulsive, and a number of periodic orbits which are hyperbolic; but our v is Morse-Smale, with no cycles. For simplicity, let us assume that we have only two hyperbolic periodic orbits O_3 and O_4, with O_3 dominating O_4, that is the unstable manifold of O_3 intersects the stable manifold of O_4 and not vice-versa. We want to define the hybrid object M/v.

For this, we consider the trace of the stable manifold of O_3 on the boundary ∂T_1 of a torus T_1 transverse to v around O_1.

If the eigenvalues of the Poincare-return map at O_3 are negative, then the trace of $W_s(O_3)$ on ∂T_1 has only one connected component.

If the Poincare-return map at O_3 has positive eigenvalues, then there are exactly two connected components to this trace because the stable manifold of O_3, when deprived of O_3, has two connected components and they do not intersect. Each of these is an embedded differentiable closed curve. By standard arguments, it follows that either both curves are embedded isotopic closed curves which both read homotopically as $ma+nb$ on the two S^1-generators of the fundamental group of ∂T_1; or one or both of them are contractible to a point in ∂T_1.

Let us assume that we are in this second case: the results which we derive then can be adapted to the first case.

Let us think of the intersection of the stable manifold of O_4 with ∂T_1. This intersection, though an embedded differentiable curve, is neither closed nor compact. It could also not be connected We claim that it is made of a finite number of connected components, each of them being an embedded differentiable closed curve whose closure is obtained by addition of one or both connected components of $W_s(O_3) \cap \partial T_1$. This is a fine point which we need to understand.

Let us also assume for simplicity that none of the components of the trace of $W_s(O_3)$ on ∂T_1 is homotopic to zero in ∂T_1.

Assuming in the sequel that both components of $W_s(O_3) \cap \partial T_1$ are not homotopic to zero, they both read $ma+nb$, m, n prime to each other (they are then homotopic since they do not intersect). The components of $W_s(O_4) \cap \partial T_1$ then "spiral" towards these two isotopic embedded curves.

1.$\hat{c}$ and the fundamental domain.

We consider a section $\hat{c}$ to $W_s(O_3) \cap \partial T_1$ in ∂T_1. This section is made of two small embedded pieces of curve defined on two intervals, which are transversal to each of the components of $W_s(O_3) \cap \partial T_1$ and which are connected by two other embedded pieces of curves in $\partial T_1 \setminus (W_s(O_3) \cup W_s(O_4)) \cap \partial T_1$. We find a closed differentiable embedded curve transverse to both traces of $W_s(O_3)$ and $W_s(O_4)$ on ∂T_1.

We now consider the Poincare-return map f of v from a section σ to v near O_1 containing $\hat{c}$. σ needs not be transverse to v at O_1, but it should be everywhere else. The choice is very clear if, denoting b the generator transverse to O_1, m in the couple (m, n) defining the homotopy class of $W_s(O_3) \cap \partial T_1$ is non-zero. We can take for σ a disk transverse to O_1 in T_1.

We assume, without loss of generality, that $f(\hat{c})$ is in σ. $f(\hat{c})$ is drawn on the boundary of the solid torus $f(T_1)$. f is generated by the one parameter group of v, γ_s and we thus can write $f = \gamma_{s(.)}$, where $s(.)$ is an appropriate function. We can consider the family of tori $\gamma_{ts(.)}(T_1)$, $t \in [0, 1]$. They define a family of curves in σ which define a fundamental domain Δ. We iterate this fundamental domain Δ under positive and negative powers of f. The negative iterations end at O_1. The positive iterates go where they should go, but we are going to track a few portions of Δ under positive iterations.

Observe that $\hat{c}$ intersects each of the components of $W_s(O_3) \cap \partial T_1$ at exactly one point. This point, under positive iterations, will get closer and closer to O_3. Adjusting f nearby O_3 to become the Poincare-return map of O_3 at one of its points z, in an appropriate section σ_1, a portion of Δ defined by two small transversals in $\hat{c}$, $f(\hat{c})$ containing the points of $W_s(O_3)$ in these sets (there are two of them in each of $\hat{c}$, $f(\hat{c})$) and two other "vertical " pieces of curves connecting these two couples of points(we thereby find a small "rectangle" in σ) will reach σ_1 under iteration from **both** "sides". Indeed, the "vertical"curves connecting the points of $W_s(O_3)$ under (adjusted) iteration will reach O_3 on two distinct sides, thus z in σ_1 from two distinct sides. Just as in [8], in Morse Theory, when considering a non-degenerate critical point, these two "vertical" transversals then "spread" under iteration along $W_u(O_3) \cap \sigma_1$ and its iterates. we will denote this set $\tilde{W}_u(O_3)_z$. It is clear that we have to add it to $\cup f^n(\Delta)$ in order to define M/v.

We now have to evolve to O_4 from O_1 and from O_3. We may assume that the two small transversals in $\hat{c}$ to $W_s(O_3)$ contain all of $W_s(O_4) \cap \hat{c}$ and thus that $\hat{c}$ outside of these small transversals "spouses" $W_s(O_4) \cap T_1$ without intersecting it. Thus, the iterates under f of $W_s(O_4) \cap \hat{c}$ all go to σ_1 and, from there, should go to O_4.

This situation, the lower stage of the tower of domination, offers a new background which we want to discuss now: the first case is when O_3 dominates O_4 and does not dominate any other hyperbolic orbit. We first elaborate more on this specific situation. We then consider the case when O_3 can dominate more than one, typically two-the arguments then generalize-hyperbolic periodic orbits.

Let us discuss the first case. Let T be the Poincare-return map of O_4, defined on a section σ_4 to O_4, at a point of O_4. T is generated by the one-parameter group of v and therefore T is homotopic to the identity map, in the set of invertible two dimensional maps. Thus, the differential of T at the origin has a positive determinant. O_4 is hyperbolic, thus the differential of T has two real eigenvalues, of the same sign, one larger than one in absolute value, the other one less than one in absolute value also. The differential of T^2 has only positive eigenvalues. Let us consider $W_s(O_3)\cap\sigma_1$, which is made of one interval, two intervals I^+, I^-, after removing the fixed point. These two intervals are part of two half-lines L^+, L^- which span through the use of the one parameter group of v all of $W_u(O_3)$ (after the addition of O_3). We can imagine that σ_1 has been extended via the use of the Poincare-return map of O_3 and then the time $1; t$-map of v so that it reaches near O_4 and "touches" σ_2 (σ_1, after iterations, and σ_2 have to intersect since O_3 dominates O_4).

2. How O_1 dominates O_3: a special case.

Let us assume that σ_1 **and** σ_2 can be built as small hyperbolic neighborhoods of pieces of sections to v in $W_u(O_3)$ and $W_s(O_4)$ respectively. I^+, I^- and L^+, L^- have been defined for $W_u(O_3)$, but they can also be defined for O_4. We denote them J^+, J^-, P^+, P^-. We use $I^\pm$, $J^\pm$, $L^\pm$, $P^\pm$ to define these sections. The I's and J's are used when we are on the "sides" of $W_u(O_3)$ and $W_s(O_4)$ that intersect, taking I and J until a common point x which we may choose to be in σ_2, near O_4.

Observe that $W_u(O_3)$ and $W_s(O_4)$ intersect in fact at infinitely many points, a subset of which is derived from x through the use of T.

We then claim-and this claim is more important for the complete understanding of this specific configuration rather than for the definition of M/v-:

Proposition 1. *Assume that σ_1* **and** *σ_2 can be built as small hyperbolic neighborhoods of pieces of sections to v in $W_u(O_3)$ and $W_s(O_4)$. Then, there is another set of points of intersection generated by a y which is not derived from x by iterations. In fact, the intersection points can be viewed as couples of such points (x, y) together with their iterates.*

Proof of Proposition 1. Indeed, considering T^2, we know that its differential has positive eigenvalues at zero. thus T^2 maps J^+ into J^+ and J^- into J^- respectively. Without loss of generality, we may assume that it is J^+ and I^+ that intersect at x. We consider then $W_u(O_3)\cap\sigma_2$ and more specifically the subset corresponding to I^+.

Part of $\hat{c}$ was made of two small pieces of curves transverse to $W_s(O_3)\cap\partial T_1$. we also considered $f(\hat{c})$, thus the image under f of these two small pieces of curves and we connected the ends of each corresponding pair of intervals by "vertical" lines; this yields two pairs (V_1, V_2) and (V_3, V_4) of "vertical lines" which we iterate using f. The related lines are again denoted V_i. Each V_i is as close as we please to the "history", under iteration", of one of the two points of $\hat{c}\cap(W_s(O_3)\cap\partial T_1)$. This "history" defines a set of lines in $W_s(O_3)\cup W_u(O_3)\cup W_s(O_4)\cup W_u(O_4)$, in fact in the intersection of these sets with the respective sections σ, σ_j. each V_i neither intersects $W_s(O_3)$, nor $W_s(O_4)$.

The smaller fundamental domain defined by the two small "vertical" lines connecting the two intervals and their images under f spread under iteration and "fill" σ_1 (up to the addition of $W_u(O_3)\cap\sigma_1$) and, from there, they "spread" until they "touch" σ_2 just as σ_1 did. From there, we use T and we move to σ_2, which we can view to be bounded on each side by portions of the V_is. Under T^2, the I^+-portion of $W_u(O_3)\cap\sigma_1$ maps into a half-line. Indeed, the image curve does not intersect the V_is, because, if it did, then some points of V_i would not,under reverse iteration, go to O_1, but would go to O_3. It is therefore entirely contained into σ_2. If the differential of T at the origin has positive eigenvalues, then we can use T in lieu of T^2.

This half-"line" intersects J^+, that is the portion of $W_s(O_4)\cap\sigma_2$, into at least one point, namely x,hence also at its iterates under T^2. Under iteration, it "spreads" and its tangent direction becomes pore and more parallel to $W_s(O_4)\cap\sigma_2$.

If the differential of T at zero has negative eigenvalues, then $T^{2k+1}(x)$ is in J^- rather than J^+ and all the points of intersection of $J^+\cap T^2(I^+)$ would then read as $T^{2k}(x)$ if x and only x spans this intersection. However, the orientation of $T^2(I^+)$ alternates at consecutive intersection points, going from left to right (according to a certain orientation of $W_s(O_4)\cap\sigma_2$) at a point and from right to left at the next point. These consecutive points, because x

spans the intersection set, are iterates of each other under T^2. This yields a contradiction because the eigenvalues of the differential of T^2 at zero are both positive.

The same argument works with T in lieu of T^2 if the differential of T at zero has positive eigenvalues.◀

We now have tracked our fundamental domain under evolution.we have understood how one of the two sides of each of σ_1 and σ_2 are "filled" by the smaller fundamental domain under iteration. The other sides are "filled" because $\hat{c}$ intersects as well the trace of the stable manifolds of O_3, O_4 on ∂T_1 and we can "drive" M/v, with a proper choice of f to bring our set under iteration to "fill" the other sides. We add to these iterated sets the two curves $W_u(O_3) \cap \sigma_1$, $W_u(O_4) \cap \sigma_2$. The construction of M/v is nearly completed in this easier framework, when O_3 dominates only O_4. We still need to understand how this set behaves near O_1, O_2.

3. The general case.

There is a more complicated case, when O_3 dominates more than one hyperbolic orbit; e.g O_3 dominates O_4, O_5, both hyperbolic orbits. The construction of M/v is then greatly simplified by the following Proposition:

Proposition 2. *Assume N is a two-dimensional surface transverse to v and intersecting e.g $W_u(O_3)$ at a point z which is e.g on the v-orbit of a point of e.g L^+. Then $N \cap W_u(O_3)$ contains a whole half-line which is an image of L^+ through the one parameter group of v. This statement holds when N is a surface with boundary; then L^+ is replaced by an interval I^+.*

Proof of Proposition 2.

The intersection of $W_u(O_3)$ and N is a transverse intersection, which therefore yields a differentiable manifold of dimension 1 transverse to v. Let us consider the connected component of this intersection containing z. It is a one dimensional manifold that has a natural projection π over L^+/J^+. π defines a fibration because any two points of $\pi^{-1}(\ell)$, ℓ given in L^+, can never coalesce: v is transverse to this manifold. It follows that this fibration extends throughout L^+, unless it is limited by the boundary of N.◀

4. Outline of the construction of M/v.

The construction of M/v is derived from the choice of the curve $\hat{c}$ on ∂T_1 and from Proposition 2. A careful choice of $\hat{c}$ allows us to move from the attractive orbit O_1 to the other hyperbolic orbits and to the repulsive orbit (there could be more attractive and repulsive orbits; we are only describing a simple case here). As we have seen above, near a hyperbolic orbit O_3 which is "directly" dominated by O_1 (i.e there is no intermediate hyperbolic orbit), the iterates of a fundamental domain built using $\hat{c}$ and its image through the Poincare return map of O_1 will fill a suitable section of this hyperbolic orbit from "the two sides" (the process is different if the Poincare return map at O_3 has positive or negative determinant; in the first case, the trace of the stable manifold of O_3 on ∂O_4 is connected while, in the second case, it has two connected components).

The construction of M/v is therefore very clear near O_1 and from there to all such O_3s.

Then, from a hyperbolic orbit O_3, we may move to another hyperbolic orbit O_4. There, we use Proposition 2 which tells us that, because our iterations of our fundamental domain include a point of the stable manifold of O_4, they will contain all the trace of this stable manifold in an appropriate section. In this way, a full "side" of this section of O_4 will be "filled". The other side will be "filled" either through a similar process, that is starting from O_3; or directly from $\hat{c}$ because $\hat{c}$ is chosen appropriately to intersect the trace of the part of the stable manifold of O_4 that goes directly to ∂O_1.

The process continues in this way (we add to the sections which we encounter the traces of the hyperbolic orbits O_is in these sections), until we have exhausted all hyperbolic orbits. We are then left with O_2. Our fundamental domain under iterations will come to O_2 in a complicated manner, depending also on the linking number of O_1 and O_2, of O_2 with the other O_is.

This is the part of the construction of M/v that requires further study, until we know precisely what is involved in this construction and this object becomes thereby a straightforward object to use.

Despite the fact that our construction of this object is only sketched, we are going to introduce a function on the space of curves Imm^* defined in section 1 on this space and study its properties. This should lead us to a method for the computation of our homology [2], [3].

4. The functional.

The natural functional to use on the space of curves $\Lambda_{T^m}(M/v)$ defined in section 2 is the action functional $J(x) = \int_0^1 \alpha_x(\dot{x})$. This functional is invariant under T if the contact form α is "symmetric". However, the contact form built through the averaging procedure of section 2 is only nearly symmetric; it is not symmetric; and the space M/v of section 3 is very well defined only outside of the periodic orbits of v: for example, if we consider the case of the standard contact structure of S^3 and if v is a Morse-Smale perturbation of a vector-field defining a Hopf fibration in $ker\alpha$, O_2, that is the repulsive periodic orbit, is a "boundary" for M/v.

Let us consider in more details the case of the standard contact structure on S^3, with v having two periodic orbits, one attractive O_1 and the other one repulsive O_2. The fundamental observation in this easier framework is that the "symmetrized" α at points x close to O_1 and O_2 (how close depends also on N, the number of iterations of T involved in the "symmetrization" process) reads $\alpha_x = \lambda(x)\alpha_{0x}$, $\lambda(x)$ tending to ∞ as x tends to $O_1 \cup O_2$; that is the "symmetrized" α has a coefficient tending to infinity on the standard contact **form** of S^3.

1. Proposition 3.

Some more is true; namely:

Proposition 3. *v around O_1 and O_2 may be arranged so that*

i) the contact vector field $\xi(x)$ of the "symmetrized" α tends to zero in norm as x tends to $O_1 \cup O_2$.

In addition,

ii) denoting ψ be the map which assigns to a point x the next coincidence point ([1], [2]) on the positive v-orbit through x, then, after perturbation, the orbits of ξ do not connect x_0 and $\psi^j(x_0)$ for x_0 in $O_1 \cup O_2$ and for $j \in Z$.

We give below the proof of Proposition 3. Using the results of [6], properly generalized, we expect Proposition 3 to hold for every nowhere zero Morse-Smale v in $ker\alpha$. The results of [6] are about the behavior of α around hyperbolic periodic orbits of v along which $ker\alpha$ does not turn well. They of course also apply to hyperbolic orbits around which $ker\alpha$ turns well; that is, it should also be possible in such a case to build "mountains" around these hyperbolic orbits (essentially, i. of Proposition 3 should hold around hyperbolic orbits). But, we can then also hope that such "mountains" can be built around the regions where $ker\alpha$ does not turn well along v. Symmetrizing α outside of small neighborhoods of such regions, we would try to perturb such a symmetric α so that ii) of Proposition 3 would hold.

Since there is some hope that Proposition 3 generalizes, it is useful to prove that this Proposition holds in the simpler case of the standard contact structure of S^3 and to indicate, pending the complete and rigorous proof of all details, how the computation of the homology, or the existence of periodic orbits for ξ can be derived from this procedure.

Proof of Proposition 3. Given an integer N, as x moves closer to the e.g attractive orbit O_1 of v, the negative iterates of ψ are expanding maps. Assume that $ker\alpha$ rotates twice (v is a perturbation of one of the Hopf-fibrations vector-fields in the kernel of the standard contact form of S^3) along O_1, with a uniform rotation and a uniform coefficient of contraction $-1 \lneq \gamma \lneq 0$ after a quarter of a turn along O_1, so that the coefficient of contraction after a full turn is γ^4. This can be achieved after a suitable perturbation of $v, ker\alpha$ near O_1. The negative iterates of α_0, at a point x_0 of O_1 therefore read as multiples $\gamma^{-i}\alpha_0$. Their sum at the order N (from 0 to $-N$) therefore reads as $\frac{1-\gamma^{-N-1}}{1-\gamma^{-1}}\alpha_0$. The coefficient in front of α_0 tends clearly to ∞ and this fact cannot be destroyed by the contribution of the positive iterates, since ψ is contracting near O_1.

This is the basic phenomenon from which, after some additional work estimating the derivatives along ξ, $[\xi, v]$ of λ (the "symmetrized" α reads $\lambda\alpha_0$), i)follows.

For ii), we observe that, given e.g y a ξ_1-piece of orbit of a symmetric (over a limit process, outside of $O_1 \cup O_2$) contact form α_1 connecting two points x_0 and $\psi^j(x_0)$ of e.g O_1, if we perturb this symmetric α_1 in the vicinity of a point of this ξ_1-piece of orbit into another, symmetric (the symmetry is along ψ and its iterates) form α_2, there will still be, if the intersection problem satisfies the appropriate transversality conditions, in the vicinity of y a ξ_2-piece of orbit connecting two points x_1 and x_2 of O_1, one close to x_0, the other one to $\psi^j(x_0)$; but, generically, x_2 will not be $\psi^j(x_1)$. ii) follows.◀

2. M^*/v and the symmetric α_s.

We now consider the case of a more general nowhere zero vector-field v in the kernel of α, which we assume to be Morse-Smale, having a number of periodic orbits $\cup O_i$, some elliptic, the other ones hyperbolic. Using the Propositions and the results of the previous section, we define the space M/v. We can also define the space:

$$M^*/v = M/v \setminus \cup O_i$$

The averaging procedure for α can be completed on M^*/v.

If we start from a point of $M \setminus \cup O_i$, the positive and negative iterates under ψ^2 (the transport map along v mapping a point x_0 to the next **oriented** coincidence point (see [1], [2]) on the v-orbit through x_0) of a given point end up near the attractive and repulsive orbits of v. It follows that "averaged" limit forms of α, α_s are well defined point-wise on M^*/v, but might have, as one can easily see, discontinuity points along the stable and unstable manifolds of the hyperbolic periodic orbits of v. If there are no such hyperbolic orbits, as in the case of the standard contact structure of S^3, with v a small perturbation of a vector-field in $ker\alpha$ defining a Hopf fibration, then a symmetric form α_s is well-defined and continuous, differentiable on $M \setminus \cup O_i$.

This contact form α_s can be used to define a symmetric functional J_s, that is a symmetrized version of J (see section 1) on the curves of Imm^* which do not intersect $\cup O_i$.

We need therefore to understand the behavior of this functional on the curves of this set that are in the immediate vicinity of curves intersecting $\cup O_i$. Typically, we would want that such curves are "far" from being critical points of J_s and we would in fact want more: namely, we would want that the functional J_s tends to ∞ as we approach such curves.

3. J_s near the hyperbolic orbits.

For hyperbolic orbits (and this also should solve the discontinuity issues involved by the hyperbolic orbits in the definition of α_s), we have devised a construction in [6]. This construction was carried around hyperbolic orbits having the property that $ker\alpha$ does not rotate well along them. Using large amounts of rotation near the attractive and repulsive orbits, one could build a contact form in the same contact structure such that its associated contact vector-field became tiny near these orbits. This construction can be carried out around the other hyperbolic orbits as well, that is around the orbits along which $ker\alpha$ turns well. It should imply that a functional J can be built on Imm^*, extending J_s. J should be very large or ∞ on the curves of Imm^* entering and exiting a small neighborhood of a hyperbolic orbit.

4. Three additional observations.

There are three additional observations that are useful:

First, this procedure should work around the regions of M where $ker\alpha$ does not rotate well along v. The hope is that a construction similar to the one introduced in [6] for the corresponding hyperbolic orbits can be extended to this framework.

Second, i) of Proposition 3 above implies that the functional J should be very large or ∞ on a curve of Imm^* that enters a given neighborhood of an attractive or repulsive periodic orbit of v, then intersects this periodic orbit, then exits this given neighborhood (that is J_s should tend to ∞ as we approach such a curve).

Third, ii) of Proposition 3 should also generalize into the statement that there are no curve made of pieces of orbits of the symmetric ξ_s up to v-jumps between points x and $\psi^2(x)$ intersecting at least one O_i (repulsive, attractive, or hyperbolic). This is a weaker result than the results foreseen above which say that J is very large or ∞ at curves crossing $\cup O_i$. But it should be a useful additional result.

This provides a very rudimentary version of a scheme in order to compute the homology defined in [3], [4], [7]. But, to the least, one can see here a program; and a glimmer of a reasonable hope that the non-compactness issues can be overcome in Contact Form Geometry.

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