HOMOLOGY COMPUTATION

Abbas Bahri

Let M^3 be a three dimensional, orientable compact manifold without boundary and let α be a contact form on M^3. Choosing a vector-field v in $\ker\alpha$ and using the "dual" form $\beta = d\alpha(v,.)$, we have considered in earlier works [1], [2], [3], [4] the variational problem defined by the functional $J(x) = \int_0^1 \alpha_{x(t)}(\dot{x}(t))dt$ on the space of variations $C_\beta = \{x \in H^1(S^1, M)\text{such that } \beta(\dot{x}) = 0 \text{ and } \alpha(\dot{x}) = \text{a positive constant}\}$.

This variational problem has to be set up properly. This requires some work which we will not discuss here. We will summarize in[6] the body of hypotheses underlying this approach and the family of conjectures under which this setting acquires full generality, see [5], section 4 for preliminary discussions of this issue.

Let us assume for simplicity that β is also a contact form with the same orientation than α. Then this variational problem is well defined, its critical points are the periodic orbits of the Reeb vector-field of α, which we denote ξ.

A homology can be defined using the **flow-lines** originating at these periodic orbits see [3], [4], see in particular [3],page 23 for the statement of the hypothesis **(A3)**-again we refer to [6]for the discussion of **(A3)** and all other assumptions .

In [4], we have established that most of the flow-lines of this homology were **compact**, see Theorem 1 of [4] which meant that they did not connect periodic orbits to asymptots but only to other periodic orbits. We have conjectured that all of them were compact. This is specific of the flow-lines involved in the definition of this homology and probably does not hold for all flow-lines originating at a periodic orbit. We have assumed in order to derive this result Hypotheses (A) and (B), see [4]-we restate Hypothesis (A) below. These hypotheses will also be discussed in [6].

S^1 acts on C_β.

We identify in this paper, using the Fadell-Rabinowitz cohomological index [7] of the space C_β, the generators of this homology for odd indexes.

The result contains two parts: the direct part, see section 2 of the present work, where we establish that the unstable manifold of a periodic orbit of odd index $2k+1$ generates, after removal of the curves of A^+, these are the curves such that the v-component b of their tangent vector does not change sign,the pull-back of the $2k$-dimensional generator of the real cohomology of the classifying space $E_{S^1} \to B_{S^1}$ to the principal bundle $C_\beta \to C_\beta/S^1$.

This direct part has an heuristic converse which we establish in section 3. Namely, we consider the "bad" spaces- "bad" because they might be not separated- M/v obtained by identification of points on the same v-orbit and its loop space $\Lambda(M/v)$.

We also "average" α over a large family of coincidence points (coincidence points are points on the same v-orbit such that $ker\alpha$ rotates $k\pi, k \in Z$ between them in the v-transport),we thus derive an "almost" symmetric α_0 and an associated functional

J_0. Because of the "symmetry", one can think of J_0 as defined on $\Lambda(M/v)$. Critical points are then periodic orbits of the "symmetric" α_0 or "geodesics" for the Finsler metric defined by α_0 on the "bad" space M/v. While M/v and $\Lambda(M/v)$ can be "bad" spaces, the difference of topology at the crossing of such a "geodesic" and its critical level are well defined.

Thinking of the lifts to C_β, we find on one hand that these "geodesics" have a Maslov index and on the other hand, we find that the difference of topology in C_β above them is complicated. After exploration, see Proposition 6 in section 3, we find that for S^1-bundles over M/v the difference of topology (expressed as pairs of sets) for C_β is the difference for $\Lambda(M/v)$ multiplied by S^1 and that in all other cases, the differences of topology in C_β and in $\Lambda(M/v)$ were the same.

We exclude- for simplicity- the case of S^1-bundles. We then prove that if the Fadell-Rabinowitz index of the level sets of J_0 changes at the crossing of the critical level of the "geodesic", this "geodesic" is of Maslov index zero and lifts to a periodic orbit.

This falls short of a computation of the homology in two regards: first, we still have to prove that C_β is for example of infinite Fadell-Rabinowitz index. This holds true on S^3 for example see [2], p268. The more general setting where this method acquires full generality requires to consider the space C_β only on part of the underlying manifold M, after removing some curves or points see [5], section 4, for a preliminary result. A similar result would have to hold in this setting (at least a result stating that this index changes along the level sets).

Second, the converse part of this result is heuristic exactly in measure of how bad our spaces near the "geodesics", how "symmetric" an α_0 we can build etc, we have made this entirely explicit.It ultimately reduces to the understanding of the behavior of $ker\alpha$ on the ω-limit set of v. For example,if we could find a v in $ker\alpha$ such that $ker\alpha$ "turns" well along v see [1] pp24-25 - this means that every point has at least a coincidence point for positive times and for negative times- outside of a finite set of (we may assume hyperbolic) periodic orbits of v and non degenerate zeros of v, we believe that this heuristic part can be made entirely rigorous.

Before proceeding with the outline and the main body of this work, we point out the following basic facts:

All curves x of C_β have a tangent vector which can be decomposed into $\dot{x} = a\xi + bv$. A key result in this theory [2], [3] states that there is a decreasing pseudo-gradient for J on C_β which never increases the number of zeros of b. This result is assumed and used throughout [3], [4], [5]. It is as well used here.

Another consequence of the existence of this pseudo-gradient is that the set A^+ of curves such that b does not change sign and is not identically zero is invariant under this decreasing flow. This is also a basic fact used in this paper.

Furthermore, the asymptotes of this pseudo-gradient have been studied in detail in [2], [3]. These asymptotes correspond to curves y_∞ made of pieces of ξ-orbits alternated with pieces of $\pm v$-orbits, The $\pm v$-jumps obey various constraints [3]. An unstable manifold $W_u(y_\infty)$ is associated to these critical points at infinity. The maximal number of zeros of the v-component b for a curve on this unstable manifold is denoted $\theta(y_\infty)$. This number never increases as we move along the decreasing pseudo-gradient from a critical point (at infinity)to another critical point (at infinity) which it dominates (i.e there is a decreasing flow-line from the first one to the second one).

We proceed now with the outline of the paper:

Section 1 is devoted to state and prove several particular results needed for the homology computation:

We study in particular the set of curves x of C_β which are near the level zero of our functional. For a given integer k_0, we consider a subset of this set characterized by the requirement that the v-component of the tangent vector to a curve of this set should have at most $2k_0$ zeros. We estimate the Fadell-Rabinowitz cohomological index of such a subset.

We then make several observations about the set A^+ of curves of C_β such that their v-component b does not change sign. We introduce in this framework a hypothesis, Hypothesis (C) which is entirely not needed in this work but renders the proofs and the statements of the results easier.

We conclude section 1 with a study of the unstable manifold of some special critical points at infinity of J, see [3], which have a single $\pm v$-jump. Given such a critical point at infinity y_∞ of index 2ℓ, we can modify, using the results of [3] pp 80-102, α in its vicinity so that $\theta(y_\infty) = 2\ell - 2$.

In section 2, we consider a periodic orbit of odd Morse index and after manipulating its unstable manifold and removing from it the functions of A^+, we introduce a (relative) cycle of C_β.We prove that the integral on this cycle of the S^1-Chern class of the appropriate dimension is non zero. This describes the change of topology associated to a periodic orbit of odd Morse index whenever this periodic orbit contributes to a non zero cycle in the homology which we have defined in [3].

Section 3 is devoted to establish a "converse" to this result, hence a characterization of these cycles. we introduce the "bad"spaces M/v, $\Lambda(M/v)$, we "symmetrize" α into α_0. The variational problem corresponds then (in an extended sense) to the problem of finding "geodesics" for the Finsler metric defined by α_0 on M/v. We then prove that if the Fadell-Rabinowitz index of the level sets of the functional changes, the lift to M of the critical point is a closed periodic orbit of the Reeb vector-field of α_0.

We proceed now with the main body of the paper:

1. On the Fadell-Rabinowitz cohomological index of certain sets.

We consider the curves of $\overline{C}_\beta$ which have $a = 0$, thus $\dot{x} = bv$. Equivalently, this set may be described as J_0, the level zero of the functional J extended to $\overline{C}_\beta$. These curves are drawn on v-pieces of orbits. We consider in the sequel the component of of J_0 where

$$\int_0^1 b = 0.$$

We will more specifically consider, for $k_0 \in \mathbb{N}$, the set:

$$\mathcal{M}_{k_0} = \{x \in \overline{C}_\beta; \dot{x} = bv, \int_0^1 b = 0, b \text{ has at most } 2k_0 \text{ zeros }\}.$$

Clearly, for a curve $x \in \mathcal{M}_{k_0t}$, the function b , which is the v-component of $\dot{x}$, has at least two zeros and at most $2k_0$ zeros. Thus denoting p_{k_0} the L^2-orthogonal projection onto $Span\{cos2\pi jt, sin2\pi jt, j = 1, \cdots, k_0\}$,$p_{k_0}(b)$ is non zero. This map is S^1-invariant. Thus:

Proposition 1. *$\mathcal{M}_{k_0}$ is of Fadell-Rabinowitz index at most $k_0 - 1$*

Next, we come back to the set A^+. As pointed out in the introduction, this set is invariant through the decreasing pseudo-gradient Z of [2]. By the results of [3], we may also assume that the stable and unstable manifolds of the various critical points (at infinity) are transversal.

Considering a stratified set S^+ in A^+, we claim that:

Proposition 2. *Let $y_{(\infty)}$ be a critical point(at infinity)such that either its $\pm v$-jumps do not all have the same orientation. Or, if these $\pm v$-jumps have the same orientation and if $y_{(\infty)}$ is a critical point at infinity, we assume that its H^1_0-index is non zero; if $y_{(\infty)}$ is a critical point, we assume that it is not a minimum.*

Then, S^+ does not dominate $y_{(\infty)}$(there is no flow-line from the first to the second one)

Proof. Would S^+ dominate $y_{(\infty)}$, then $X = W_u(y_{(\infty)})$ would be in the closure of $W_u(S^+)$. Thus, X would be contained in $\bar{A}^+$. This contradicts the assumptions □

We thus see that the critical points at infinity of A^+ which are dominated by A^+ are very constrained. They are even more constrained and their existence seems even less likely if we recall the following assumption which we have introduced in [4]. In order to state this assumption,let us recall that given a contact form and a vector-field of its kernel, we are thinking here of β and ξ, the kernel of the contact form(represented then by v) rotates monotonically along the orbits of the vector-field in the kernel (here ξ) see [1],Proposition 9 p24. We also recall that in [3], we have devised a method, given a critical point at infinity y_∞ and considering its non degenerate ξ-pieces (those along which the H^1_0 linearized problem is non degenerate, see [3]), to redistribute the v-rotation along these ξ-pieces among them. The process might create new critical points at infinity, but they will have at least one more degenerate ξ-piece than y_∞.

Hypothesis (A) then states that , when the number of non degenerate ξ-pieces of y_∞ tends to ∞, the amount of v-rotation (counted after redistribution in terms of added H^1_0-index) tends to ∞.

The above statement is slightly different from the statement in [4], but the two statements are actually equivalent.

If we assume Hypothesis (A), either a critical point at infinity has a finite number of ξ-pieces equal to k_0. If all of them are of H^1_0-index equal to zero(strict index for characteristic pieces, see[3]), then their total index is finite since their index at infinity cannot exceed $2k_0$ ([3]).

If the number of ξ-pieces of this critical point at infinity tends to infinity, then using Hypothesis (A), the total v-rotation tends to infinity. We can expect after redistribution that the critical point at infinity has some non zero H^1_0-index.

We therefore introduce the following hypothesis:

Hypothesis (C) Assume that y_∞ is a critical point at infinity with all $\pm v$-jumps having the same orientation. As the Morse index of y_∞ tends to infinity, its H^1_0-index becomes non zero.

A stronger, but more convenient hypothesis states that:

Hypothesis (D) Every y_∞ with all its $\pm v$-jumps having the same orientation is of non zero H^1_0-index.

We will be providing the proofs of our statements below assuming that Hypothesis (D) holds. Let us indicate here the modifications needed in order to derive similar results under Hypothesis (C). Let us also indicate how we can get rid of both hypotheses.

Under Hypothesis (D), we can flow backwards in time curves of A^+ which are in $\bar{W}_u(T)$, that is in the closure of the unstable manifold of a set T. Let us assume that b does not keep a constant sign on all the curves of T. As we flow back a curve of $A^+ \cap \bar{W}_u(T)$, either we go back to T, which means that eventually at some time on the flow-line, we leave A^+. Or we end up at a critical point(at infinity)$y_{(\infty)}$ dominated by T. We can scale our backwards deformation so that we stop the process once x has entered a bit the set where b has at least two zeros.

If $y_{(\infty)}$ has a sign change in its $\pm v$-jumps and after flowing back, we find that our curves are near y_∞, then again we must have left A^+ at some time. Otherwise, under Hypothesis (D), if $y_{(\infty)}$ is a critical point at infinity with all its $\pm v$-jumps having the same orientation, then y_∞ is of non zero H_0^1-index. If we remove $A^+$from our sets-it is invariant by the decreasing flow- then the unstable manifold of y_∞ is deprived from a connected cone which is invariant by the decreasing flow. The remaining set can be retracted by deformation onto a single flow-line which it contains. We can arrange our deformation so that this flow-line does not go through y_∞. We thus see that under Hypothesis (D), $W_u(T)$ can be retracted by deformation onto the union of a set of curves x such their v-component has at least two zeros and at most as many zeros as the v-component of the curves of T with periodic orbits dominated by T or critical points at infinity having at least two sign changes and at most as many sign changes in their $\pm v$-jumps as b for the curves of T.

Under Hypothesis (C), the argument is slightly modified:

The previous argument holds true outside of a finite dimensional CW-complex L_+built with the unstable manifolds of the critical points at infinity with H_0^1-index zero and all $\pm v$-jumps having the same orientation. L_+ is entirely contained in A^+.

Because L_+ is finite dimensional, the homology and cohomology of index large enough of the pair $C_\beta/S^1, L_+/S^1$ is the same than that of C_β/S^1. This holds true also for the Fadell-Rabinowitz cohomological index. The pull-back of the S^1-Chern class taken at a large power will be zero in the cohomology of L_+. If it is non zero in the cohomology of C_β/S^1, then it can be traced back to the cohomology of the pair $(C_\beta/S^1, L_+/S^1)$.

Let $\mathcal{V}$ be an S^1-equivariant neighborhood of L_+. Because there is an equivariant map from $\partial\mathcal{V}$ to L_+, the Fadell-Rabinowitz index of $\partial\mathcal{V}$ is less than or equal to the one of L_+. For this reason, all the computations with the Chern classes elevated at a high power can be carried in $C_\beta - \mathcal{V}$ and they can be traced to the cohomology of the pair $\left(((C_\beta - \mathcal{V})/S^1, \partial\mathcal{V}\right)/S^1$ which is the same than the one of the pair $(C_\beta/S^1, L_+/S^1)$. The computations of section 2, Propositions 3 and 4 below will be carried out in C_β/S^1 deprived of $\mathcal{V}/S^1$.

If we remove now Hypotheses (C) and (D), we can still work on the variational space $C_\beta - A^+$. We then introduce an S^1-equivariant neighborhood $\mathcal{V}$ of A^+ and work with the cohomology and the pull-back of the Chern classes to $(C_\beta - \mathcal{V})/S^1$ on one hand and with the homology of the pair $\left((C_\beta - \mathcal{V})/S^1, \partial\mathcal{V}/S^1\right)$ or $\left((C_\beta - \mathcal{V})/S^1\right)/\partial V/S^1)$.The same results would hold.

A basic question then remains: under minimal conditions, it is easy to see that the Fadell-Rabinowitz cohomological index of C_β is infinite, see[2], p268. Is the

Fadell-Rabinowitz index of $C_\beta - A^+$ infinite?

We conclude section 1 with the following observation , which will be used in section 3, about critical points at infinity x_∞, of H_0^1-index i_0 and index at infinity i_∞-see [3]- having a single **non-degenerate** ξ-piece and thus a single $\pm v$-jump. We claim that:

Lemma 1. *α can be perturbed with a C^2-bounded, C^1-small perturbation near the ξ-piece of x_∞ so that the maximal number of zeros of b on $W_u(x_\infty)$, $i_0+\gamma$ equals $i_0+i_\infty-2$ if i_0+i_∞ is even, larger than or equal to 4.*

Proof. Set $i_0+i_\infty = 2k$. By the results of [3], we know that we can perturb α with a C^2-bounded, C^1-small perturbation so that i_0 achieves two other consecutive values besides the original one. Since i_0 is, in all cases, at most $2k$, we can change i_0 so that it is equal to $2k-2$. Then $i_0+\bar{\gamma}$ is also $2k-2$ □

As we proceed with the transmutation of x_∞, we create pairs $(\bar{z}, \bar{z}')$ of false critical points at infinity having one characteristic piece. We are changing the H_0^1-index i_0 of x_∞ by **lowering** it. Using the results of [1], p134, we can claim:

Lemma 2. *Neither $\bar{z}$ nor $\bar{z}'$ is of index $2k$*

Proof. Assume that the transmutation involves the collapse of x_∞ and $\bar{z}$. For x_∞, the index "at infinity" i_∞- see [3]- increases by 1 throughout the transmutation(s) which it incurs. For a given transmutation of this type, $\bar{z}$ and $x\infty$ exchange their indexes "at infinity"-i.e their indexes in the Γ_{2s}'s, these spaces are the spaces of curves made of as many ξ-pieces and $\pm v$-pieces as our critical points at infinity, here $s=1$, they are manifolds of dimension $2s$, see [2]- through the collapse. Thus, after the collapse, the index "at infinity" of $\bar{z}$ becomes i_∞ while the index of x_∞ changes from i_∞ to $i_\infty+1$. Lemma 2 follows.

As a critical point at infinity, $\bar{z}$ has to be taken with its strict H_0^1-unstable manifold because it has a single (characteristic) $\xi - piece$. If the H_0^1-index of x_∞ is i_0, so is the full H_0^1-index of $\bar{z}$: they collapse.

Hence, the total index of $\bar{z}$ after the collapse is $i_0-1+i_\infty \neq 2k$.

As for $\bar{z}'$, its index "at infinity" remains throughout the transmutation $i_\infty+1\pm1$, because it was created in a canceling pair with $\bar{z}$ which was of index "at infinity" $i_\infty+1$ before the collapse with x_∞. Its strict H_0^1 index is i_0-1 also. Therefore its total index is

$$i_0 - 1 + i_\infty + 1 \pm 1 = i_0 + i_\infty \pm 1 \neq 2k.$$

Our claim follows □

2. On the S^1-Chern classes of C_β/S^1.

Consider a periodic orbit of index $2k+1$, x_{2k+1}.

Its unstable manifold contains a part where b does not change sign. Under the hypothesis stated above, this part does not dominate any critical point or critical point at infinity besides the orbits of v at the bottom level. We can then flow back all this part to x_{2k+1}. This defines a global retraction by deformation which ends into the closure of the subset of $W_u(x_{2k+1})$ generated by the orthogonal of the first eigenfunction (least eigenvalue) in the negative eigenspace of the second derivative for x_{2k+1}. We may view this subset as generated by a disk D^{2k} in the unstable disk of x_{2k+1}.

Assume now that $\partial x_{2k+1} = 0$. ∂ is here the boundary operator in our homology see [3], [4]. This is equivalent to say that the usual boundary operator of Morse theory-however extended to include the asymptotes- takes its values in critical points (which are necessarily critical points at infinity) of index $2k$ such that the maximal number of zeros of b on their unstable manifold is at most $2k-2$.

For simplicity, we assume below that the usual boundary operator applied to x_{2k+1} yields zero.

This assumption is equivalent, after adjusting the intersection operator-this does not change the maximal number of zeros of b beyond $2k$-to considering a general cycle generated by a combination of periodic orbits of such index.

Let S^{2k-1} be the boundary of D^{2k}. Let f be the classifying map for the S^1-action defined by (see [4]):

$$f(x) = b^+ \int_0^1 b^- - b^- \int_0^1 b^+$$

where $\dot{x} = a\xi + bv$. This yields the following commutative diagram:

$$\begin{array}{ccc} C_\beta^* & \xrightarrow{f} & E_{S^1} \\ \downarrow & & \downarrow p \\ C_\beta^*/S^1 & \xrightarrow{\tilde{f}} & B_{S^1} \end{array}$$

where

$$C_\beta^* = \{x \in C_\beta \text{ s.t } b \not\equiv b^+ \text{ and } b \not\equiv -b^-\} \cap H^2$$

and E_{S^1} is viewed here as $E - \{0\}$ where E is the infinite dimensional space $H^1(S^1, \mathbb{R}) - \{0\}$.

For x in S^{2k-1}, $f(x)$ has at most $2k$ zeros and at least 2 zeros. Then, $p_k \circ f(x)$ is non zero. p_k is the orthogonal projector onto $\underset{j=1}{\overset{k}{Span}}\{cos2\pi jt, sin2\pi jt\}$.

$p \circ f(S^{2k-1})$ is easily deformed onto $p \circ p_k \circ f(S^{2k-1})$.

The image can be parametrized using sections to the S_1 action on the equivariant sphere $\frac{p_k \circ f}{|p_k \circ f|_{L^2}}(S^{2k-1})$. This gives rise to a cycle in B_{S^1} which can be identified as the generator of the rational homology of $P\mathbb{C}^{k-1}$, w_{k-2}.

We then have the diagram:

$$\begin{array}{ccc} S^{2k-1} = \partial D^{2k} & \xrightarrow{g_k = p_k \circ f / |p_k \circ f|_{L^2}} & S^{2k-1} \\ Id \downarrow & & \downarrow p \\ S^{2k-1} & \xrightarrow{\tilde{f}} & \mathbf{P}\mathbb{C}^{k-1} \end{array}$$

$g_k = p_k \circ f /_{|p_k \circ f|_{L^2}}$ is of degree 1.

Let θ be the generator (via cup product) of the real cohomology of $P\mathbb{C}^\infty$. Let us consider the chain of E_{S^1} equal to $c_1 = p \circ f(D^{2k})$. Observe that D^{2k} is transverse to the S^1-action (as well as S^{2k-1}). We can adjust f so that $p \circ f(D^{2k})_{|\partial S^{2k-1}} = p \circ \left(\frac{p_k \circ f}{|p_k \circ f|_{L^2}}\right)_{|\partial S^{2k-1}}$. c_1 is a cycle since $f(S^{2k-1})$ is an equivariant sphere which dimension collapses when we mod out by the S^1-action.

Let us compute

$$\int_{c_1} \theta^k = \int_{p\circ f(D^{2k})} \theta^k = \int_{D^{2k}} (p\circ f)^*\theta^k$$

Since $f(D^{2k})$ has an equivariant S^{2k-1} as chain-boundary. Therefore,

$$\int_{c_1} \theta^k = 1.$$

Thus

$$\int_{D^{2k}} (p\circ f)^*\theta^k = 1.$$

Let now y or y_∞ be a critical point (at infinity) dominated by D^{2k}. y or y_∞ is then of index $2k-1$ at most. Assume that this index is $2k-1$. For a periodic orbit of index $2k-1$, the maximal number of zeros on its unstable manifols is $2k-2$.

There are strong restrictions on the critical points at infinity that D^{2k} could dominate . Indeed, adapting the proof of Compactness [4] to the present framework, we can claim under Hypothesis (A) and any of the Hypotheses (B) of [4] that the number of $\pm v$-jumps of such a critical point at infinity y_∞ is bounded depending only on v and α, provided that the maximal number of zeros of b on its unstable manifold is $2k$. Thus, as k tends to infinity, the H^1_0-index of some of the ξ-pieces tend to infinity.

Full compactness, which we conjecture ([4],[6]) would get rid of these y_∞'s, but our results do not allow us to claim this yet. However such a y_∞ can be of one of two types: either the basic $\pm v$-jumps of y_∞ bear a sign change. These y_∞'s together with their unstable manifolds build a set

$$L = \{\cup W_u(y_\infty); y_\infty \text{of index } 2k-1 \text{ dominated by } D^{2k},$$
$$\text{maximal number of zeros of } b \text{ on } W_u(y_\infty) = 2k\}$$

Or all the $\pm v$-jumps of y_∞ have the same orientation, e.g positive. Since the H^1_0-index of some of the ξ-pieces of y_∞ is non zero (Hypothesis (D) or k large), the curves on $W_u(y_\infty)$ directed along the positive eigenfunctions of the linearized operator $\ddot\eta + a^2\eta\tau$ under the various Dirichlet boundary conditions corresponding to its various ξ-pieces (at first order $\delta b = \ddot\eta + a^2\eta tau$, see [4]) have also all their $\pm v$-jumps positively oriented. They are in A_+. In fact $A_+ \cap W_u(y_\infty)$ can then be described as a cone of flow-lines around these curves.

The curves which belong to the closure of this set are contained in the unstable manifolds of critical points of J in A_+. Under Hypothesis (D), they are themselves of non-zero H^1_0-index. Thus the above argument repeats: an entire C of flow-lines, which includes these special critical points at infinity, is thereby defined. This cone has no intersection with D^{2k} and is contained into the set (with ε designating a small positive number):

$$A_{2k-2} = (\{x \in C_\beta \text{such that } b \text{ has at most } 2k-2 \text{ zeros or } b \equiv 0\} \cap \{x \in W_u(y_{(\infty)}, y_{(\infty)}$$
$$\text{s.t associated maximal number of zeros of } b \text{ is at most } 2k-2\}) \cup \{x \in C_\beta \cap J_\varepsilon,$$
$$\text{such that } \int_0^1 b \text{ is close to zero and } b \text{ has at most } 2k \text{ zeros}\},$$

Proposition 4, which we prove below, involves this cone. However, its claim is a homological statement involving a pair of spaces, the smaller of both containing the cone. We therefore, along the above arguments, will be excising C from this pair of sets.

We first have the following straightforward result:

Proposition 3. $\overline{W_u(D^{2k})}$ *defines a cycle in* $H^{2k}(C_\beta/S^1, (A_{2k-2} \cup L)/S^1)$

S^1 acts effectively on C_β so that the classifying map f extends in fact to C_β:

$$\begin{array}{ccc} C_\beta & \xrightarrow{\ f\ } & E_{S^1} \\ \downarrow & & \downarrow p \\ C_\beta/_{S^1} & \xrightarrow{\ \tilde{f}\ } & B_{S^1} \end{array}$$

and we can assume that $\tilde{f}_{|\partial D^{2k}} = f_{|D^{2k}} = \frac{p_k \circ f}{|p_k \circ f|_{L^2}}$.

We now claim:

Proposition 4. $\tilde{f}^*\theta^k$ *is zero in* $H^{2k}\left((A_{2k-2} \cup L)/_{S^1}, Q\right)$ *and defines therefore a cocycle of* $H^{2k}\left(C_\beta/_{S^1}, (A_{2k-2} \cup L)/_{S^1}\right)$ *which the image of a cocycle in* $H^{2k}\left(C_\beta/_{S^1}, A_{2k-2}/_{S^1}\right)$. *Furthermore,*

$$\int_{\overline{W_u(D^{2k})}} \tilde{f}^*\theta^k = 1.$$

This computation holds in the duality H^{2k}/H_{2k} *for the pair of sets* $(C_\beta/_{S^1}, (A_{2k-2} \cup L)/_{S^1})$.

Proof. A_{2k-2} is made of two parts; one part is A^*_{2k-2} where b changes sign and another part A^+ where b does not change sign. A^*_{2k-2} is also made of two parts, a part over which b has at most $2k-2$ zeros (and at least two) and another part, near the bottom level, where b has at least two zeros and at most $2k$ zeros. We denote in the sequel $A * S^1$ the equivariant set derived by S^1 action on the set A. We claim that

$$\Delta_k = A_{2k-2} \cup \left((W_u(D^{2k}) - D^{2k}) * S^1\right) \cup L$$

has a Fadell-Rabinowitz cohomological index [4] less than or equal to $k-1$. In fact, we claim a much stronger statement, namely that there is an S^1 equivariant map from $\Delta_k/(\Delta_k/S^1)$ into $S^{2k-1}/P\mathbb{C}^{k-1}$ which is the restriction to $\Delta_k/(\Delta_k/S^1)$ of the classifying map $f/\tilde{f}$ for the S^1-action on C_β. This claim holds under Hypothesis (D) and has to be slightly modified to obtain full generality by removing A^+ from all the sets considered.

It is important here to emphasize what part of the bottom level J_0 to include in Δ_k. We do not assert this claim if we include the periodic or recurrent orbits of v in Δ_k. However, the claim holds if we include the part of J_0 made of curves drawn onto v-pieces of orbits which are contractible and such that b has at most $2k$ zeros.

Let us assume that we have established this claim and let us proceed with the proof of the proposition.

First, let us consider a chain c in $S_{2k}((A_{2k-2} \cup L)/S^1)$ and compute the value of $\tilde{f}^*\theta^k$ on it. We find:

$$\tilde{f}^*\theta^k(c) = \theta^k(\tilde{f}_*(c)) = \int_{\tilde{f}(c)} \theta^k$$

$\tilde{f}(c)$ is of dimension $2k-2$, we thus find that these integrals are zero. The first claim of the Proposition follows.

Let us now consider $\partial D^{2k} = S^{2k-1}$, the unstable sphere of x_{2k+1} and let us consider $S^{2k-1} * S^1$, the S^1-invariant set derived from S^{2k-1} through time-translation.

$W_u(S^{2k-1} * S^1) \cup A_{2k-2} \cup L$ retracts by deformation equivariantly onto $A_{2k-2} \cup L$ because every flow-line out of $S^{2k-1} * S^1$ ends either at $y_{(\infty)}$ and near $W_u(y_{(\infty)}$ with $y_{(\infty)}$ of index $2k-1$, this is contained in A_{2k-2}; or such a flow-line ends near the bottom level, then it is either in A^+ or it is in A_{2k-2} because it is near the contractible curves drawn on pieces of v-orbits and b has at most $2k$ zeros. We have used here the assumption $\partial x_{2k+1} = 0$ or the more general form of this assumption.

Since $\Delta_k = W_u(S^{2k-1} * S^1) \cup A_{2k-2} \cup L$ retracts by deformation equivariantly onto $A_{2k-2} \cup L$, $\tilde{f}^*\theta^k \in H^{2k}(C_\beta/S^1, \Delta_k)$.

$W_u(D^{2k})$ is a cycle in the chain group $S_{2k}(C_\beta/S^1, \Delta_k)$ which is as well represented by D^{2k}.

Let us assume, arguing by contradiction, that D^{2k} is homologically zero. Then

$$D^{2k} = \partial c + a,$$

where a is a chain valued into Δ_k with boundary $\partial a = S^{2k-1}$. Using our claim above, $\tilde{f}(a)$ is contained in $P\mathbb{C}^{k-1}$. Accordingly,

$$\tilde{f}^*\theta^k(a) = \theta^k(\tilde{f}_*(a)) = \int_{\tilde{f}(a)} \theta^k = 0$$

since $\tilde{f}(a)$ is of dimension $2k-2$ at most. On the other hand, since θ^k is closed,

$$\tilde{f}^*(\theta^k)(\partial c) = \theta^k(\tilde{f}_*(\partial c)) = \theta^k(\partial \tilde{f}_*(c)) = \int_{\partial \tilde{f}(c)} \theta^k = 0.$$

Thus, $\int_{D^{2k}} \tilde{f}^*\theta^k = 0$, a contradiction.

Let us now prove our claim about Δ_k and its classifying map:

Δ_k is made of four parts: a part contained in $W_u(S^{2k-1})$ where b has at least two zeros and at most $2k$ zeros, A^*_{2k-2}, A^+, and the critical points (at infinity) y, y_∞(the maximal number of zeros of b on their unstable manifolds is at most $2k-2$, see above). A^+ cannot, under Hypothesis (D), dominate any critical point (at infinity). We flow back the part of A^+ which is contained in $W_u(S^{2k-1})$ until we either enter into the set of curves with a v-component b having at least two zeros or we reach a neighborhood of a critical point (at infinity) which is necessarily dominated by x_{2k+1} and cannot be x_{2k+1} by construction of D^{2k}. We flow back as well along the same deformation all the curves of A^+ which all belong to the unstable manifold of a critical point (at infinity) with maximal number of zeros $2k-2$ or less. This defines an equivariant retraction by deformation on a new set Δ^1_k.

On the part of Δ^1_k where b has $2k$ zeros, the classifying map can be viewed as provided by $g_k = \frac{p_k \circ f}{|p_k \circ f|_{L^2}}$, where p_k is the L^2-orthogonal projection onto $Span\{\cos 2\pi i x, \sin 2\pi i x, i = 1, .., k.\}$. This map becomes $g_{k-1} = \frac{p_{k-1} \circ f}{|p_{k-1} \circ f|_{L^2}}$ as we move to the part of Δ^1_{2k} where b has at least two zeros and at most $2k-2$ zeros.

In a neighborhood of the critical points(at infinity), the classifying map is valued into S^1.

We need to glue these two maps, this is usually completed by defining a product map over the intersection of the two domains into the product space $\mathbb{C}^k - \{0\} \times \mathbb{C} - \{0\}$. This would lead us into $\mathbb{C}^{k+1} - \{0\}$, one dimension too much. We thus need to be able to replace the map g_k by g_{k-1}. Observe that the critical points (at infinity) which are involved here are of two types: either, the maximal number of zeros of b on their unstable manifold is at most $2k-2$; or the basic $\pm v$-jumps of these critical points at infinity bear a sign-change. We have removed the part in A^+. Thus, the classifying map can be reduced to g_{k-1} outside a neighborhood of the critical points (at infinity). This extends by continuity to a neighborhood of this set. We extend this map to a neighborhood of the critical points at infinity using the product map described above. We obtain in this way a map valued into $\mathbb{C}^k/\mathbb{PC}^{k-1}$ which we denote $\bar{h}/h$. $\bar{h}$ is defined on a neighborhood of the set formed of the union of the curves x such that b has at least two zeros and at most $2k-2$ zeros with a neighborhood of the critical points (at infinity) discussed above. On the boundary of this neighborhood (among the curves such that b has at most $2k$ zeros), g_k is another classifying map. We need to glue them because g_k is known to extend while we do not know the same fact for $\bar{h}$. Observe that g_k and $\bar{h}$ coincide outside a neighborhood of each critical point (at infinity) $y_{(\infty)}$ (which we know to be of index $2k-1$ or less and with a maximal number of zeros equal to $2k$ or less in its unstable manifold). If $y_{(\infty)}$ is a genuine critical point at infinity, then either its $\pm v$-jumps do not have the same orientation, g_k works throughout, i.e we can take $\bar{h}$ to be g_k also near y_∞. Or all the $\pm v$-jumps of y_∞ have the same orientation. If the H_0^1-index of y_∞ is non zero, then y_∞ does not hinder the downwards deformation after removal of the functions of A^+ and of the cone C defined above, before stating Proposition 3. We do not have to consider such y_∞'s. Hypothesis (D) rules out the other critical points at infinity. We are left with the periodic orbits of index $2k-1$, thus after the removal of the x's of A^+ with a disc D^{2k-2} and the set it forms under S^1 action, $S^1 * D^{2k-2}$. $\bar{h}$ and g_k coincide on $S^1 * \partial D^{2k-2} = S^1 * S^{2k-3}$. $\bar{h}$ extends to $S^1 * D^{2k-2}$. Do does g_k if we perturb a bit D^{2k-2} so that it does not contain the periodic orbit anymore. The two maps are S^1-equivariant, in order to glue them we just need to show that their restriction to $\{1\} \times D^{2k-2}$ are homotopic as maps valued into S^{2k-1} relative to their common boundary value. Using a dimension argument, this conclusion is obvious and the claim follows. The proof of Proposition 4 is complete

Let $\tilde{A}_{2k-2}$ be the set of curves $\{x \in C_\beta$ such that b has at most $2k-2$ zeros or $b \equiv 0$, number of zeros of b on W_u(periodic orbit) at most equal to $2k-2\}$. Using the variational flow of [2], $\tilde{A}_{2k-2} \cup L \cup J_0$ retracts by deformation onto $A_{2k-2} \cup L \cup J_0$. Using excision, we derive:

Proposition 5. *D^{2k} defines a non zero cycle in $H_{2k}(C_\beta \cup J_0, \tilde{A}_{2k-2} \cup L \cup J_0)$, "dual" to $\tilde{f}^*\theta^k$*

Observation: Let $S^1 * x_{2k}$ and $S^1 * D^{2k}$ be a critical circle of strict index $2k$ and its unstable manifold. We can try to repeat the argument. However, now, after removing $b \equiv b^+$ or $b \equiv b^-$, we find $S^1 * S^{2k-2}$ which we map using f into $Span\{\cos 2\pi jt, \cos 2\pi jt, j = 1, \cdots, j = k\} - \{0\}$. We may consider that $f_{|S^1 * S^{2k-2}}$ is valued into the equivariant unit sphere S^{2k-1}. However, f can be checked to be of zero degree now. The argument collapses. Also, the quotient is now (D^{2k-1}, S^{2k-2}) which is not of even dimension. Thus $f^*\theta^k$ is achieved dually by x_{2k+1} and not by

$S^1 * x_{2k}$.

3. To what extent is the converse true?.

In the previous two sections, we have studied and characterized the homology class of C_β/S^1 which is associated to a periodic orbit of odd Morse index. This statement falls short from a variational characterization of some periodic orbits since we have to start with a cycle associated to an odd index in the homology which we have defined in [3], [4].

We explore now how much the converse holds. The approach will lead us to define some objects and sets closely related to the Fredholm problem in the variational problem (J, C_β).

We need for our discussion to introduce two spaces which are "bad" because their topology is not separated in general and also a (almost) symmetrized form of our contact form α. The discussion can then start.

The first "bad" space is the space of orbits M/v where points on the same v-orbits are identified. This is of course in general a space where points are not separated. The second space is even worst than this space since it is the space of loops $\Lambda(M/v)$ of M/v. There are natural projections $M \to M/v$ and $C_\beta \to \Lambda(M/v)$ although we are bypassing here regularity questions. The second projection is S^1-equivariant so that the classifying map for the S^1-action on C_β factorizes through $\Lambda(M/v)$, $f^*\theta^k$ which we discussed above may be viewed as an image of a cocycle of $H^{2k}(\Lambda(M/v)$.

On another hand, let us consider the contact form α. Let ψ be the map defined through v-transport which associates to every given point x the next coincidence point x_1 (i.e α rotates 2π from x to x_1 along the v-orbit through x, see [1],p25 for more details). Let us assume for sake of simplicity that every x of M has one coincidence point, this issue is discussed in [6]. Then every x have infinitely many positive and negative iterated coincidence point. Every iterated coincidence point of order i defines a function $\lambda_i(x)$ which is the collinearity coefficient of the transported α from x_1 to x onto α. Picking up a large number N, averaging the value of a collinearity coefficient over λ_1, λ_N, we can build a new form $\tilde{\alpha}$ which is almost symmetric, at least if we compare through v -transport the value of the form between coincidence points corresponding to orders od iterations which are $o(N)$. We are assuming here a slow growth of the $\lambda_i' s$ with i.

For the sake of simplicity, let us assume that we actually have a "symmetric" $\tilde{\alpha}$ i.e $\psi^*\tilde{\alpha} = \alpha$.

Considering then a "tangent " vector z to M/v, we can lift it to M so that it coincides in direction, transversally to v, with $\tilde{\xi}$ which is the Reeb vector-field of $\tilde{\alpha}$. Because of the symmetry, we derive in fact a countable number of lifting positions in M. However, denoting $\bar{z}$ one of the lifts, $\tilde{\alpha}(\bar{z})$ is independent of the lifting position. Thus, J can be viewed as a functional $\tilde{J}$ defined on $\Lambda(M/v)$.

We can look for critical points of $\tilde{J}$ on $\Lambda(M/v)$, all of this is completed in a generalized sense for the moment, we will make sense of this approach later. We can look in particular for critical points which correspond to changes in the Fadell-Rabinowitz cohomological index of the level sets of $\tilde{J}$. We would expect those to lift into periodic orbits.

This conclusion is actually not so obvious. Indeed, let us compare at the crossing of a critical value, which we assume to correspond to a single critical point, the change of topology in the level sets of J with the change of topology in the level sets of $\tilde{J}$. Let c be the critical value, $\bar{x}$ be the critical point, D^l be its unstable disk.

Then, the pair $(\tilde{J}_{c+\varepsilon}/S^1, \tilde{J}_{c-\varepsilon}/S^1)$ is homologically equivalent to $(D^l, \partial D^l)$ which makes perfect sense.

We would like to understand the pair $(J_{c+\varepsilon}, J_{c-\varepsilon})$. This pair is more complicated to understand because $\bar{x}$ could first lift into a curve which is not closed if it is of non zero "Maslov" index, that is what we would like to rule out with the particular change of topology sought, related to the index γ_{FR}. Furthermore, above $\bar{x}$, in C_β, there are other curves, other critical points at infinity containing $\pm v$-jumps between conjugate points i.e coincidence points such that $\lambda_i = 1$.

Would the problem be reduced to the non zero "Maslov" index and the lift contain a single curve, we could rule it out using Proposition 5 above. But in fact there are several lifts, a continuum of them in our framework, because of the symmetry and there are on top of that all the critical points at infinity. Is it still the Chern class which we are obtaining above?

In order to understand this issue, let us describe the critical set at infinity above $\bar{x}$: let x be a smooth lift of $\bar{x}$, x may not be closed if it is of non zero Maslov index. Its edges are then conjugate (in the present symmetric framework, coincidence)points. Let i be an index which is $o(N)$. $\psi^i(x)$ is also a lift of $\bar{x}$ and, in a generalized sense, so are all the curves y which are partially drawn on a $\psi^i(x)$, then on a $\psi^j(x)$ etc, the partial trajectories being completed along $\tilde{\xi}$, until the full interval of time $[0, 1]$ is completed. The transitions from $\psi^i(x)$ to $\psi^j(x)$ are completed along $\pm v$-jumps between conjugate points. This is a huge critical set K which has transversally the same Morse index (this statement is not obvious, see [2],[3]).

Thus, $H^*(J_{c+\varepsilon}/S^1, J_{c-\varepsilon}/S^1)$ equals $H^*(K \times (D^l, \partial D^l))$. Thus, we can reformulate our search as follows: What is the topology of K?

In order to answer this question, we need to extend a bit K as follows: K is built out of pieces of the $\psi^j(x)$ which are connected by $\pm v$-jumps. The pieces patch to cover $\bar{x}$.

If we break a little bit the symmetry, we introduce characteristic surfaces Σ_i which are defined by the equation $\lambda_i(x) = 1$. We have described in [3],p9 and Chapter IV how, given a piece of ξ-orbit which runs into $\{x/\lambda_i(x) > 1\}$, we can then create an "oscillation" along v at one of the points of this piece of ξ-orbit in order to reach the coincidence point of order i, then open up this "oscillation" at this coincidence point, insert a piece of ξ orbit, the whole curve closes and J decreases, see [3] pp 38-40 and 158-161, also [2] pp 236-244.This is due to the lack of Fredholm behavior of this variational problem.

Accordingly, above $\bar{x}$, after this perturbation which breaks the symmetry is completed, we have an optimal lift $\tilde{x}$. Indeed, above each point of $\bar{x}$, there is (under some conditions about the behavior of the λ_i's as i tends to ∞) an optimal i. $\bar{x}$ lifts near this point along this determination until the lift hits a characteristic surface and we have to switch the determination. The curve $\tilde{x}$ obtained in this way is a fixed point for a map Φ, easy to define as it runs along ξ from a characteristic surface to the next one (on the "right" side), completing $\pm v$-jumps between conjugate points as the curve hits each hypersurface. This fixed point can be assumed to be non degenerate.

Assume that v does not define an S^1-bundle over M/v. Then, starting from a point of K, we cannot reach a corresponding point of $\tilde{x}$ using a +/-oscillation along v as well as a -/+ one. One of these oscillations would not work. The deformation onto $\tilde{x}$ introduced above is thus well defined; the set on which it is defined is a slight extension of the set K; this extension $\tilde{K}$ allows for the building up of oscillations along v. The related downwards deformation after the introduction of the perturbation ($\tilde{K}$ can be kept unchanged through the perturbation since $\bar{x}$ can be assumed to be non degenerate) deforms all of $\tilde{K}$ over $\tilde{x}$. Some work [6] shows that the deformation is compatible with the (small) unstable manifold D^l.

Thus, assuming that v does not define an S^1-bundle over M/v and that no periodic orbit of v passes through a point of $\tilde{x}$, we expect the following result to hold:

Proposition 6. $H^*(J_{c+\varepsilon}/S^1, J_{c-\varepsilon}/S^1) = H^*(D^l, \partial D^l) = H^*(\tilde{J}_{c+\varepsilon}/S^1, \tilde{J}_{c-\varepsilon}/S^1)$

Observation: in the case of S^1-bundles generated by v, it is natural to expect a slightly modified result, namely that $\tilde{K}$ will deform onto a circle S^1 rather than on the single point defined by $\tilde{x}$.

We now "prove"(we have indicated that some of our spaces are "bad") that in this unchanged framework(symmetry,exclusion of S^1-bundles), any change in the Fadell-Rabinowitz index of the level sets of $J/\tilde{J}$ is achieved by curves $\bar{x}$ of Maslov index zero.

Let r be the projection map from C_β/S^1 to $\Lambda(M/v)/S^1$ and let $\tilde{g}/g$ be the classifying map for the S^1-action over $\Lambda(M/v)$. θ denotes as above the S^1-Chern class.

We assume in the sequel that $\omega^k = (r \circ g)^*(\theta^k)$ is non zero in $H^{2k}(J_{c+\varepsilon}/S^1)$ and that it is zero in $H^{2k}(J_{c-\varepsilon}/S^1)$. Thus ω^k generates $H^{2k}(J_{c+\varepsilon}/S^1, J_{c-\varepsilon}/S^1)$.

We are assuming that we have symmetry. Arguing by contradiction, let x^∞_{2k} be an associated critical point at infinity with exactly one $\pm v$-jump. Using ψ^j, we can generate the other ones (having exactly one $\pm v$-jump).

We can use Lemma 1 above and derive, after averaging, a symmetric α such that the maximal number of zeros of b on the unstable manifold of x^∞_{2k} and the like is $2k-2$ at most. This unstable manifold may be represented using $(2k-2)*$'s at various locations. Near x^∞_{2k}, one of them corresponds to the large $\pm v$-jump of x^∞_{2k}. Obviously, in this representation, the location of the $\pm v$-jumps even near x^∞_{2k} can change to account for two additional directions. We then claim:

Lemma 3. *∂x^∞_{2k} is valued in $A_{2k-4} \cup \{$periodic orbits or critical points at infinity of index $2k-2$ at most and their unstable manifolds$\} \cup C \cup D$ where C is contained in $\{x \in J_\epsilon, \int_0^1 b$ is close to zero and b has at most $2k-2$ zeros$\}$ and $D = \{\cup W_u(x^\infty_{2k-1}), x^\infty_{2k-1}$ having at most one characteristic ξ-piece of large H^1_0-index $(\geq m_0, m_0$ a fixed integer) ; maximal number of zeros of b on $W_u(x^\infty_{2k-1}) = 2k-2\}$* ∎

Proof. x^∞_{2k} cannot dominate a periodic orbit of index $2k-1$ because by [3], Lemma 3 p80, the maximal number of zeros on its unstable manifold should be at least $2k$

Considering a critical point at infinity (to pursue this argument, we can introduce a small perturbation destroying the symmetry) of index $2k-1$, either the maximal number of zeros on its unstable manifold is $2k-2$ and this critical point at infinity contains two characteristic pieces of large enough H^1_0 index. Invoking as above a natural extension of the arguments of Compactness [4], we can infer that the intersection number of x^∞_{2k} with this critical point at infinity is zero. Or this critical point at infinity contains no characteristic piece of H^1_0-index $\geq m_0$. Using then Hypothesis (A) of [4], we can change the maximal number of zeros of b on its unstable manifold by 2. The only case left is the case when x^∞_{2k-1} has a single characteristic piece of large enough index. We have conjectured in [4] that we could get rid of such curves in the framework of flow-lines originating at periodic orbits. This conjecture extends here to the x^{∞}_{2k-1}'s dominated by such x^{∞}_{2k}'s, as in Lemma 3. Short of assuming this conjecture, we have to add the set D in the statement of Lemma 3.

L_{2k-2} designates in the sequel a stratified space of dimension $2k-2$ at most.

Let us now find a converse to the characterisation of the previous section. Indeed, instead of working with the level sets of J, we could work with the cohomology group of order $2k$ of the pair $(J_d/S^1, (J_d \cap (A_{2k-2} \cup L_{2k-2})/S^1)$. As the Fadell-Rabinowitz index of the set J_d becomes $2k$, the cocycle θ^k is non zero, after pull-back, in the cohomology of J_d/S^1, but it is zero in the cohomology of $(J_d \cap (A_{2k-2} \cup L_{2k-2}))/S^1$. Thus it can be traced back to $H^{2k}(J_d/S^1, (J_d \cap (A_{2k-2} \cup L_{2k-2})/S^1)$.

Since A_{2k-2} is invariant by our decreasing flow, the cohomology of order $2k$ of the pair (L_{2k-2} does not count) is non zero at a critical(at infinity) level c such that the Fadell-Rabinowitz cohomological index of the level sets increases.

In the framework defined above, which excludes S^1-bundles, the change of topology in the level sets of J can be expressed using a single x^{∞} and its unstable manifold. The optimal $\tilde{x}^{\infty}$ obtained using the characteristic surfaces Σ_i is the natural choice for x^{∞}, but any x^{∞} of the family will in fact work.

We can choose among all possible choices an x^{∞} such that the number of zeros of b on its unstable manifold is minimal. Usually, the number of $\pm v$-jumps of such an x^{∞} has to be minimal since in this symmetric framework, the H^1_0-decreasing variations adjust to the maps ψ^i: if we have a decreasing direction z along x^{∞} and if we consider a curve y^{∞} obtained after transporting a piece of ξ-curve of x^{∞} along ψ^i, then the direction z' obtained by pushing z through ψ^i over the same time span is also decreasing for y^{∞} now.

The number of zeros of b for the curve "$y^{\infty} + \varepsilon z'$" can only grow when compared to the number of zeros of b for "$x^{\infty} + \varepsilon z$".

We thus can choose x^{∞} to be the critical point (at infinity) in the family having the least number of $\pm v$-jumps, one at most. While the existence of x^{∞} is clear once the symmetry is broken, its uniqueness is less clear as well as the possibility of reaching x^{∞} without introducing new zeros for b.

We will not try to complete a deformation in the case of zero Maslov index, in fact we think that it might be wrong in this case. But in the case of nonzero Maslov index, the curves having a single $\pm v$-jump (without base point) form a line. It is on this line that the maximal number of zeros of b is minimal i.e starting from any curve of K, we can reach a curve on this line without ever increasing the maximal number of zeros of b on the unstable manifolds of the critical curves through which we deform. Thinking of this line now, we break the symmetry and we create a unique global minimum which also achieves the minimum in the maximal number of zeros of b. Up to cancelations of local maxima and minima of J along this line, we can deform all of them onto the global minimum. The only worry which we should have is with the maximal number of zeros of b which might jump by 2 along the deformation. But this can be adjusted using the techniques of [3] p81: we have a piece of ξ-orbit which spans the entire piece of ξ-orbit of the global minimum $\tilde{x}$ and the entire piece of ξ-orbit of $\psi(\tilde{x})$. On each piece of ξ-orbit, we have adjusted the v-rotation along ξ, it spans $2k-2$ nodes. We then can turn the v-rotation to occur at constant speed along ξ over the two pieces of ξ-orbits together. This is where we use the techniques of [3] p81. Using this determination to build a symmetric contact form around the line, we now can state that all along this line of curves of non zero Maslov index, the maximal number of zeros of b on the unstable manifolds is $2k-2$. Breaking the symmetry and arguing as above, we conclude that:

Proposition 7. . *Let c be the critical level of $\tilde{x}$ which we assume to be of non zero Maslov index. We assume c to be isolated. $H^{2k}(J_c, J_c \cap (A_{2k-2} \cup L_{2k-2}))$ is then zero.*

Corollary 1. *A level c at which the Fadell-Rabinowitz index of the level sets J_d changes corresponds in the symmetric framework (excluding S^1-bundles generated by v) to critical curves of Maslov index zero (hence to periodic orbits).*

Observation 1. We expect a similar result to hold for S^1-bundles.

Observation 2. The homology of [3], [4] is invariant, under suitable hypotheses, under deformation of the contact form. Thus, the above result derived in the symmetric case, should in fact be a general result on the homology (given the vector-field v).

Let us point the following byproduct result which we can derive from our argument above:

Lemma 4. *Assuming the same hypotheses than in Proposition 7 and assuming that we can get rid of the set D in the statement of Lemma 3, $H^*(J_{c+\varepsilon}, J_{c+\varepsilon} \cap A_{2k-4}) = H^*(J_{c+\varepsilon}, J_{c-\varepsilon} \cap A_{2k-4})$ for ε positive small enough.*

Proof. The maximal number of zeros on the unstable manifold of $\tilde{x}$ is exactly $2k-2$. Therefore, $J_{c+\varepsilon} \cap A_{2k-4}$ can be moved down, past the level c

We thus have concluded in this paper that at a crossing of a critical point at infinity of index $2k$ and of a non zero Maslov index, the cohomology of the pair $(J_d/S^1, (J_d \cap (A_{2k-2} \cup L_{2k-2})/S^1)$ could not change since, by Lemma 1, we can always choose α in the vicinity of x_{2k}^{∞} so that the maximal number of zeros on its unstable manifold is $2k-2$. We thus have developed a characterization of the difference of topology related to the changes in the S^1-Fadell-Rabinowitz index of the level sets of the functional J: these should all be related to the existence of periodic orbits of odd index, at least in the "symmetric" case, probably in a more general framework. The description of our results is now complete.

References

1. Bahri, A., *Pseudo-Orbits of Contact Forms*, Pitman Research Notes in Mathematics Series No. 173, Longman Scientific and Technical, Longman, London, 1988.
2. Bahri, A., *Classical and Quantic periodic motions of multiply polarized spin-manifolds.*, Pitman Research Notes in Mathematics Series No. 378, Longman and Addison - Wesley, London and Reading, MA, 1998.
3. Bahri, A., *Flow-lines and Algebraic invariants in Contact Form Geometry PNLDE* **53** (2003), Birkhauser, Boston.
4. Bahri, A., *Compactness*, Rutgers preprint (2007).
5. Bahri, A., Xu, Y, *Recent Progress in Conformal Geometry*, Imperial College Press 007.
6. Bahri, A., (to appear).
7. Fadell, E, Rabinowitz, P.H., *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems*, Invent.Math **45** (1978), 139–174.

RUTGERS, THE STATE UNIVERSITY OF NEW JERSEY