ON THE DYNAMICS OF A CONTACT STRUCTURE ALONG A VECTOR FIELD OF ITS KERNEL

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Introduction and Statement of Results

Let M^3 be a three dimensional manifold and α be a contact form on M. Let v be a vector-field in $\ker\alpha$ which we assume throughout this work to have a finite number of non degenerate periodic orbits and also a finite number of circles of zeros.

Let us consider near a point x_0 of M a frame (v, e_1, e_2) transported by v. $\ker\alpha$ defines a trace in Span (e_1, e_2) generated by $u = \alpha(e_2)e_1 - \alpha(e_1)e_2$. The fact that α is a contact form translates into a property of monotone rotation of u along v-transport, see [1] Propostion 9 p 24 for more details. Thus, given a point $y_0 \in M$ and the v-orbit through y_0, there is a definite amount of rotation of $\ker\alpha$ on the positive v-orbit and on the negative v-orbit.

It is natural to ask whether these amounts are infinite and the answer to this question is negative since one can produce (see [1 Section 12]) non singular codimension 1 foliation transverse to contact structures. If v generates the intersection of the tangent plane to the foliation with the kernel of the contact structure, the amount of rotation has to be less than π on any positive or negative v-orbit.

On another hand, having an infinite amount of rotation for all half v-orbits can be quite useful: introducing the dual form $\beta = d\alpha(v, \cdot), \alpha$ and β are transverse, both have v in their respective kernels. If $\ker\alpha$ rotates infinitely along v, so does $\ker\beta$. $\ker\beta$ could have some reverse rotations but it must essentially be a contact form with the same rotation than α.

When β is a contact form with the same rotation than α, a very interesting framework sets in: we introduce the space $\mathcal{L}_\beta = \{x \in H^1(S^1, M) \text{ s.t. } \beta_x(\dot{x}) \equiv 0\}$ of Legendrian curves of β and also the more constrained space $C_\beta = \{x \in \mathcal{L}_\beta \text{ s.t. } \alpha(\dot{x}) = \text{a positive constant}\}$.

On C_β, the action functional $J(x) = \int_0^1 \alpha_x(\dot{x})dt$ has the periodic orbits of ξ, the Reeb vector-field of α, as critical points (of finite Morse index).

The variational problem is not compact (there are asymptotes) but one can nevertheless, after the construction of a special flow [3], define a homology related to the periodic orbits of ξ [3]. It is therefore interesting to establish, given a contact structure α, that one can find a vector-field v in $\ker\alpha$ such that the amount of rotation on each half-orbit is infinite.

We consider hence vector-fields v in $\ker\alpha$ which have an ω-limit set reduced to their periodic orbits and their circles of zeros i.e. essentially Morse-Smale-type vector-fields (with lines of zeros allowed). Near the attractive orbit of v, after possibly perturbing slightly, v, an infinite amount of rotation is warranted on all half v-orbits attracted by this periodci orbit of v. There is a similar statement for the repulsive periodic orbit.

On the other hand, one can produce models of hyperbolic periodic orbits for v and models of contact forms having v in their kernel such that the amount of rotation of $\ker\alpha$ along these hyperbolic orbits is finite.

This type of orbits is called in this paper "bad hyperbolic orbits". They do not allow to set the variational problem J on C_β properly. Ideally, we would like to get rid of them or to the least to be able to consider the variational problem J on C_β away from these bad regions. In order to achieve this goal, a natural idea which comes to mind is to use the large rotations available (after perturbation possibly) near the attractive or near the repulsive periodic orbits. A diffeomorphism would then redistribute this large rotation over other regions of M, for example around the bad hyperbolic orbit. In this way, the bad hyperbolic orbit could be "surrounded" by a large rotation of $\ker\alpha$ along v, either coming from the attractive orbit or from the repulsive orbit.

This approach has a defect: it does not keep bounds. The price to pay for redistributing the rotation from the attractive or repulsive orbit becomes exponentially high with the amount of rotation.

The bounds carefully built [2] on the L^1-length of b as we deform curves of C_β along a pseudo-gradient for J (one of them is "curve shortening flow" which we do not use because of its "bad" behavior at blow-up) and which rely on a bound from above on $\tau = -d\alpha([\xi, v], [\xi, [\xi, v]])$ collapse.

We need therefore to find another way to introduce a large rotation. We consider two nested tori surrounding the attractive orbit for example. We introduce a second order differential equation which takes the form:

$$[v, [v, \xi]] = -\xi + \gamma(s)[\xi, v] - \gamma'(s)ds(\xi)v.$$

The unknown is ξ and the solution provides us with an extension of α. This differential equation has a unique solution under the condition that we should match α (up to a multiplicative constant) on the boundary of each torus. This differential equation, with v properly re-scaled, generates a large amount of rotation. We may introduce this rotation and keep the existing periodic orbits of ξ unperturbed. Some new ones may appear but they are precisely localized and they appear in canceling pairs. Furthermore, the bounds on all relevant quantities to the variation problem J on C_β (on $|\bar{\mu}|, |d\bar{\mu}|$ see [2] or τ from above) hold, unchanged.

This is the first part of this work. The relevant results are described in lemma 1, corollary 1, proposition 1, 2 and their corollaries. The results can be summarized in the following theorems:

Theorem 1. *Using the second order differential equation introduced above, α can be modified between two nested tori $T_1 \subset T_2$ surrounding either the attractive or the repulsive periodic orbits of v so that γ is identically zero on a smaller subinterval. Furthermore, no new orbit of ξ is introduced in the process.*

Theorem 2. *On the time interval where γ is identically zero (corresponding to $T_2 - T_1$, where $T_1 \subset T_2$ are the nested tori), α can be modified into α_N such that $\ker\alpha_N$ completes at least N full rotations on the time interval and α_N extends outside of $T_2 - T_1$ into α.*

Furthermore, after re-parameterizing v into v_N such that $d\alpha(v_N, [\xi_N, v_N]) = -1$ (here ξ_N is the Reeb vector field of α_N, this rescaling takes place outside a fixed neighborhood of the bad hyperbolic orbit),

$$\tau_N = d\alpha_N([\xi_N, v_N], [\xi_N, [\xi_N, v_N]])$$

is bounded above independent of N. Choosing a connection on M,

$$[\xi_N, \xi_N], [\xi_N, v_N], [\xi_N, v_N].\xi_N \text{ and } [\xi_N, v_N].[\xi_N, v_N]$$

are bounded transversally to v_N.

We then move, in the second part of this work, to set the variational problem J on C_β using this large rotation.

We introduce a "Hamiltonian" $\lambda = e^{\sum \theta_i \delta_i s_i}$. s_i is a measure of the rotation of $\ker\alpha$ completed at a given point on a v-orbit originating at the boundary of one of the tori. λ is localized near the stable and unstable manifold of the bad hyperbolic orbit. Replacing α by $\lambda\alpha$, we build "mountains" around the bad hyperbolic orbit i.e. regions where the Reeb vector-field of $\lambda\alpha$ is extremely small while the action is large. We prove that the bound from above still holds on τ, independently of λ and that the variational problem J on C_{β_λ} can be defined. Furthermore, we consider compact subsets of C_{β_λ} enjoying bounds independent of λ. Under decreasing deformation along the flow-lines of the pseudo-gradient, these compact sets never enter small preassigned neighborhoods of the bad orbits.

This holds in particular for all the flow-lines which start at the (unperturbed) periodic orbits of ξ. The definition of our homology follows and is independent of λ. The results of the second part which have described above can be found in Proposition 4,5,6,7 of the second part of this work entitled conformal deformation. We can summarize them in the following theorem:

Theorem 3. *λ can be built so that setting $\tilde{\alpha} = \lambda\alpha$ and re-scaling v into $\tilde{v}$ such that $d\alpha(\tilde{v}, [\tilde{\xi}, \tilde{v}]) = -1$ outside a presribed small neighborhood of the "bad hyperbolic orbit" $\mathcal{O}$, we have:*

i)$\tilde{\tau} = d\alpha([\tilde{\xi}, \tilde{v}], [\tilde{\xi}, [\tilde{\xi}, \tilde{v}]) \leq C$, where C is independent of N, independent of the amount of rotation introduced and the spreading of this rotation around the 'bad hyperbolic orbit' (embedded in the construction of λ);

ii)Considering a fixed index k_0, the periodic orbits of the Reeb vector field ξ_0 of α_0 of index k_0, k_0+1, k_0-1 (which are also periodic orbits of $\tilde{\xi}$) and their unstable manifold in C_β, the curves on these unstable manifolds do not enter a fixed small neighborhood of $\mathcal{O}$.

These results indicate that the assumption that $\beta = d\alpha(v, \cdot)$ is a contact form with the same orientation than α all over M is not needed in order to define the homology of [3].

One may question the generality of this method. We consider in this paper only the simpler case of a single bad hyperbolic orbit and we assume that its stable and unstable manifolds are caught by the attractive and repulsive orbits, in short that there is no flow-line connecting hyperbolic orbits.

However, such connecting flow-lines can easily be added, as well as circles of zero as long as they are attractive, repulsive or hyperbolic not of mixed behavior.

The only constraint lies with the hypothesis that the ω-limit set of v is made of periodic orbits and circles of zeros.

Some thought shows that this hypothesis is not needed, but it makes our study much easier. This hypothesis can be weakened and a more general behavior allowed; we expect that there is always, given a contact structure, a vector-field v in its kernel with this behavior.

We proceed now with the proof of our results.

Using the Differential Equation to Modify α

Let us consider the differential equation:

$$[v,[v,\xi]] = -\xi + \gamma(s)[\xi,v] - \gamma'(s)ds(\xi)v$$

where s is, in this first step, the time along v.

More generally, for $\varphi = \varphi(s) > 0$, we consider the differential equation:

$$[\varphi v,[\varphi v,\xi]] = -\xi + \gamma(s)[\xi,\varphi v] - \gamma'(s)ds(\xi)\varphi v \tag{*}$$

$\gamma(s)$ could be replaced by a function $\gamma(x_0,s)$ where x_0 is an initial data for the flow-line of v and s is a monotone increasing function on this flow-line. Observe that $\gamma'(s)ds(\xi) = \xi\cdot\gamma$.

Let us define α by

$$\alpha(v) = 0; \quad \alpha([\xi,\varphi v]) = 0; \quad \alpha(\xi) = 1.$$

This is possible if $v, [\xi,\varphi v]$ and ξ are independent. Writing (*) in a φv-transported frame, with $\varphi v = \frac{\partial}{\partial s_1}, \xi = A\frac{\tilde\partial}{\partial x} + B\frac{\tilde\partial}{\partial y} + C\frac{\partial}{\partial s_1}$, we derive:

$$\frac{\partial^2 A}{\partial s_1^2} + A + \gamma\frac{\partial A}{\partial s_1} = 0 \qquad \frac{\partial^2 B}{\partial s_1} + B + \gamma\frac{\partial B}{\partial s_1} = 0.$$

Thus,

$$\frac{\partial}{\partial s_1}\left(A\frac{\partial B}{\partial s_1} - B\frac{\partial A}{\partial s_1}\right) = -\gamma\left(A\frac{\partial B}{\partial s_1} - B\frac{\partial A}{\partial s_1}\right).$$

Thus, if $\varphi v, [\xi,\varphi v]$ and ξ are independent at time zero (which we will assume) they are independent thereafter.

We then have:

Lemma 1.

(i) $d\alpha(\varphi v,[\xi,\varphi v]) = -1$
(ii) $d\alpha(\varphi v,[\varphi v,[\xi,\varphi v]]) = \gamma$ *(denoted $\bar\mu$ usually)*
(iii) $[\varphi v,[\xi,[\xi,\varphi v]]] = -\gamma[\xi,[\xi,\varphi v]] + h\varphi v.$

Corollary 1.

(i) *If* $[\xi,[\xi,\varphi v]](0)$ *is collinear to* v*, then so is* $[\xi,[\xi,\varphi v]](s)$ *and* ξ *is the contact vector-field of* α*.*

(ii) *If* $\gamma=0$ *on an open set, then* $d\tau(v)=\tau_v$ *is zero on this set* $([\xi,[\xi,\varphi v]]=-\tau\varphi v)$*.*

Proof. Since $\alpha(v)=\alpha([\xi,\varphi v])=0, d\alpha(\varphi v,[\xi,\varphi v])=-\alpha\left([\varphi v,[\xi,\varphi v]]\right)=\alpha(-\xi+\gamma[\xi,\varphi v]-\xi\cdot\gamma\varphi v)=-1$

(i) follows.

Next, we observe that α is a contact form since

$$\alpha\wedge d\alpha(\varphi v,[\xi,\varphi v],\xi)=-1.$$

Let ξ_r be its Reeb vector-field. Since $d\alpha(\varphi v,\xi)=-\alpha([\varphi v,\xi])=0$ and $d\alpha(\xi,\xi)=0$,

$$\xi_r=\xi+\nu\varphi v$$

and using [2],

$$\bar\mu=d\alpha(\varphi v,[\varphi v,[\xi_r,\varphi v]])=d\alpha(\varphi v,[\varphi v,[\xi,\varphi v]])=$$

$$=d\alpha(\varphi v,\xi-\gamma[\xi,\varphi v]+z\varphi v)=-d\alpha(\varphi v,[\xi,\varphi v])\cdot\gamma=\gamma$$

(ii) follows.

Next, we compute:

$$[\xi,[\varphi v,[\varphi v,\xi]]].$$

Using (*), it is equal to:

$$\gamma[\xi,[\xi,\varphi v]]+\xi\cdot\gamma[\xi,\varphi v]-\xi\cdot\gamma[\xi,\varphi v]+h\varphi v=$$
$$=\gamma[\xi,[\xi,\varphi v]]+h\varphi v.$$

Using the Jacobi identity, it is equal to:

$$-[[\varphi v,\xi],[\xi,\varphi v]]+[\varphi v,[\xi,[\varphi v,\xi]]]=$$

$$=-[\varphi v,[\xi,[\xi,\varphi v]]].$$

Thus,

$$-[\varphi v,[\xi,[\xi,\varphi v]]]=\gamma[\xi,[\xi,\varphi v]]+h\varphi v.$$

(iii) follows.

Proof of Corollary 1. Set $\varphi v=\frac{\partial}{\partial s_1}$

(iii) reads

$$\frac{\partial U}{\partial s_1} = -\gamma U + h\frac{\partial}{\partial s_1}$$

i.e.

$$\frac{\partial}{\partial s_1}\left(e^{\int_0^{s_1}\gamma}U\right) = k\frac{\partial}{\partial s_1}$$

$$\frac{\partial}{\partial s_1}\left(e^{\int_0^{s_1}\gamma}U + \int_0^{s_1} k\frac{\partial}{\partial s_1}\right) = 0.$$

The claim follows.
Observe that

$$\alpha([\xi,\xi,\varphi v]]) = -d\alpha(\xi,[\xi,\varphi v]).$$

Set

$$\xi_r = \xi + \nu\varphi v.$$

Then,

$$-d\alpha(\xi,[\xi,\varphi v]) = -d\alpha(\xi_r - \nu\varphi v,[\xi_r - \nu\varphi v,\varphi v]) =$$

$$= \nu d\alpha(\varphi v,[\xi_r,\varphi v]) = -\nu.$$

Thus, if $[\xi,[\xi,\varphi v]]$ is collinear to v, ν is zero and $\xi = \xi_r$ □

Introducing a large rotation.

We consider now α_0 and v near the repelling (or attracting) periodic orbit of v. Their normal form, see Appendix 1, is:

$$(\alpha_0) \qquad \alpha_0 = dx + \frac{1}{20}(y+\bar{\gamma}x)d\theta \quad (\bar{\gamma} > 0)$$

$$(v) \qquad v = \left(20\frac{\partial}{\partial\theta} - (y+\bar{\gamma}x)\frac{\partial}{\partial x} + (x-\bar{\gamma}y)\frac{\partial}{\partial y}\right)\frac{1}{\sqrt{1+\bar{\gamma}^2}}.$$

Then,

$$\bar{\xi} = \frac{\partial}{\partial x} - \bar{\gamma}\frac{\partial}{\partial y}.$$

and

$$[\bar{\xi}, v] = \sqrt{1+\bar{\gamma}^2}\frac{\partial}{\partial y} \text{ so that } d\alpha_0(v,[\bar{\xi},v]) = -1.$$

Observe that $[\bar{\xi},[\bar{\xi},v]] = 0$ while

$$[v,[v,\bar{\xi}]] = -\left[20\frac{\partial}{\partial\theta} - (y+\bar{\gamma}x)\frac{\partial}{\partial x} + (x-\bar{\gamma}y)\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right] =$$

$$= -\frac{\partial}{\partial x} - \bar{\gamma}\frac{\partial}{\partial y} = -\left(\frac{\partial}{\partial x} - \bar{\gamma}\frac{\partial}{\partial y}\right) - 2\bar{\gamma}\frac{\partial}{\partial y} = -\bar{\xi} - \frac{2\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}}[\bar{\xi}, v].$$

Thus,

$$[v,[v,\bar{\xi}]] = -\bar{\xi} + \frac{2\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}}[v,\bar{\xi}] \tag{$**$}$$

$(**)$ is the same form than $(*)$.

Indeed, if we set

$$\gamma(s) = \frac{-2\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}},$$

then $\gamma' = 0$ and $(**)$ is a special case of $(*)$.

We are going to modify (**), keeping the framework of (*) but introducing a function $\gamma(s)$ which has a flat piece where it is equal to zero. We later will use this flat piece in order to introduce a large rotation of γ. However, ξ and $[\xi, v]$ are modified once γ is modified and we need to complete this modification so that the modified data for $\xi, [\xi, v]$ will glue up after a certain time $\bar{s}$ with the former data.

Observe that $\bar{\xi}$ for α_0 is in $\mathrm{Span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. Observe also that $\mathrm{Span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is invariant by the one-parameter group of v. It is easy to construct two vector-fields in $\mathrm{Span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ which commute to v. They need to satisfy:

$$\left[20\frac{\partial}{\partial\theta} - (y+\bar{\gamma}x)\frac{\partial}{\partial x} + (x-\bar{\gamma}y)\frac{\partial}{\partial y}, A_0\frac{\partial}{\partial x} + B_0\frac{\partial}{\partial y}\right] = 0.$$

This yields

$$\frac{\partial A_0}{\partial s_1} + \bar{\gamma}A_0 + B_0 = 0$$
$$\frac{\partial B_0}{\partial s_1} + A_0 + \bar{\gamma}B_0 = 0.$$

Taking $\begin{pmatrix} A_0 \\ B_0 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we derive two vector-fields $\frac{\tilde{\partial}}{\partial x}$ and $\frac{\tilde{\partial}}{\partial y}$ which have components on $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ depending only on s_1, not on the initial point (this does not hold if γ changes into $\gamma(s, x_0)$).

Since $\bar{\xi}$ for α_0 is $\frac{\partial}{\partial x} - \gamma\frac{\partial}{\partial y}, \bar{\xi}$ reads as

$$a_0\frac{\tilde{\partial}}{\partial x} + b_0\frac{\tilde{\partial}}{\partial y} \text{ with } a_0 = a_0(s_1), b_0 = b_0(s_1)$$

while

$$[v,\bar{\xi}] = \frac{\partial\bar{\xi}}{\partial s_1} = a_0'\frac{\tilde{\partial}}{\partial x} + b_0'\frac{\partial}{\partial y}\left(= \sqrt{1+\bar{\gamma}^2}\frac{\partial}{\partial y}\right)$$

Coming back to ξ and (*), with a general $\gamma = \gamma(s) = \gamma(s(s_1))$, we split ξ on the basis $\frac{\tilde{\partial}}{\partial x}, \frac{\tilde{\partial}}{\partial y}, \frac{\partial}{\partial s_1} = v$:

$$\xi = A\frac{\tilde{\partial}}{\partial x} + B\frac{\tilde{\partial}}{\partial y} + C\frac{\partial}{\partial s_1}.$$

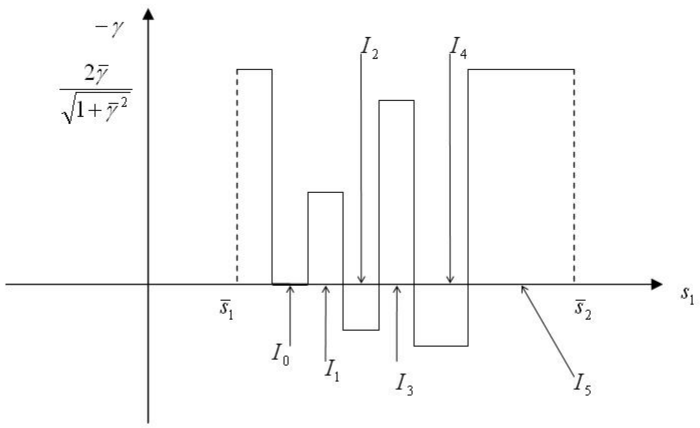
The equations satisfied by A and B are:

$$\frac{\partial^2 A}{\partial s_1^2} + \gamma\frac{\partial A}{\partial s_1} + A = 0$$

$$\frac{\partial^2 B}{\partial s_1^2} + \gamma\frac{\partial B}{\partial s_1} + B = 0.$$

We integrate these equations on an interval $[\bar{s}_1, \bar{s}_2]$, with initial data at $\bar{s}_1$ equal to $(a_0, a_0')(\bar{s}_1)$ for $(A, \frac{\partial A}{\partial s_1})(\bar{s}_1)$ and $(b_0, b_0')(\bar{s}_1)$ for $(B, \frac{\partial B}{\partial s_1})(\bar{s})$. $\gamma(s)$ near $\bar{s}_1$ is $\frac{-2\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}}$. $\left(C, \frac{\partial C}{\partial s_1}\right)(\bar{s}_1) = (0,0)$ so that ξ near $\bar{s}_1$ is $\bar{\xi}$.

$-\gamma$ will behave as follows:



Observe that we need only to worry about the components of ξ, $[\xi, v]$ on $\frac{\tilde{\partial}}{\partial x}$ and $\frac{\tilde{\partial}}{\partial y}$. Indeed, would they match as well as their s_1-derivatives with those of ξ, then ξ and $\bar{\xi}$, $[\xi, v]$ and $[\bar{\xi}, v]$ would only differ by some $\mu v, \mu_1 v$ for $s \geq \bar{s}_2$. They satisfy the same differential equation after $\bar{s}_2$. Thus α and α_0 would match also. Observe that α needs not match with α_0. It suffices that it matches with some $c\alpha_0$ (observe that $d(c\alpha_0)(v, [\frac{\bar{\xi}}{c}, v]) = -1$ and $d(c\alpha_0)(v, [v, [\frac{\bar{\xi}}{c}, v]]) = d\alpha_0(v, [v, [\bar{\xi}, v]])$. We would then extend α near the repelling (respectively attracting) periodic orbit of v with $c\alpha_0$. We need thus only to match $\xi, [\xi, v]$ and $\bar{\xi}, [\bar{\xi}, v]$ in directions with the same ratio of length (not necessarily equal to 1). We prove below that this is possible.

We have:

Proposition 1. *No new periodic orbit of ξ is created in this process.*

The proof of the above proposition requires the three following claims which follow from the construction of γ

1. γ can be constructed so that $\int|\gamma'|\leq 20$.
2. As $\bar{s}_2-\bar{s}_1$ becomes smaller and smaller, $|\gamma|$ remains bounded by 2.
3. α glues up with $c\alpha_0, c$ tending to 1 as $\bar{s}_2-\bar{s}_1$ tends to zero.

Proof of Proposition 1. As $\bar{s}_2-\bar{s}_1$ becomes small, ξ and $\bar{\xi}$ are $o(1)$-close. This is clear from the equations satisfied by the components over $\frac{\tilde{\partial}}{\partial x}$ and $\frac{\tilde{\partial}}{\partial y}$ of ξ and $\bar{\xi}$ and from the third claim stated above.

For the C-component on $\frac{\partial}{\partial s_1}$, it depends on $\gamma'(s)ds(\xi)=\gamma'(s)(Ads(\frac{\tilde{\partial}}{\partial x})+Bds(\frac{\tilde{\partial}}{\partial y})+C)$. Observe that, because $\frac{\tilde{\partial}}{\partial x},\frac{\tilde{\partial}}{\partial y}$ and $\frac{\partial}{\partial s_1}$ commute, $ds(\frac{\tilde{\partial}}{\partial y})$ and $ds(\frac{\tilde{\partial}}{\partial y})$ are independent of s_1. They are bounded uniformly. So are A and B.

Since $\int|\gamma'|$ is bounded by 20, then

$$\int_{\bar{s}_1}^{s_1}\int_{\bar{s}_1}^{\tau}|\gamma'|\leq 20\left(\sum_{j=0}^{s}|I_j|\right)=o(1).$$

After some work, this implies that the v-components of ξ and $\bar{\xi}$ are close up to $o(1)$. Indeed, C (the v-component of ξ) satisfies:

$$\frac{\partial^2 C}{\partial s_1^2}+C+\gamma\frac{\partial C}{\partial s_1}=\gamma'(s)ds(\xi)=\gamma'(s)\left(Ads\left(\frac{\tilde{\partial}}{\partial x}\right)+Bds\left(\frac{\tilde{\partial}}{\partial y}\right)+C\right)$$

while $\bar{C}$, the v-component of $\bar{\xi}$ satisfies

$$\frac{\partial^2\bar{C}}{\partial s_1^2}+\bar{C}+\bar{\gamma}\frac{\partial\bar{C}}{\partial s_1}=0$$

with $C(\bar{s}_1)=\bar{C}(\bar{s}_1)$, $\frac{\partial C}{\partial s_1}(\bar{s}_1)=\frac{\partial\bar{C}}{\partial s_1}(\bar{s}_1)$.

The claim follows.

How γ is built.

We start with the differential equation with a constant $\bar{\gamma}_0$

$$\frac{\partial^2 u}{\partial s_1^2}+\bar{\gamma}_0\frac{\partial u}{\partial s_1}+u=0.$$

We set it in a matricial form with $v=-\frac{\partial u}{\partial s}$.

Then,

$$\frac{\partial}{\partial s}\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}0&-1\\ 1&-\bar{\gamma}_0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}.$$

We claim that:

Lemma 2. *Consider with* $|\bar{\gamma}_0| < 2$*,*

$$e^{t\begin{pmatrix} 0 & -1 \\ 1 & -\bar{\gamma}_0 \end{pmatrix}}.$$

For t small, it reads up to a multiplicative factor as

$$\frac{1}{\cos\varphi}\begin{pmatrix} \cos(\beta t+\varphi) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t-\varphi) \end{pmatrix}$$

with $|\beta| < 1, \beta = \cos\varphi$*.*

Proof. Since $\bar{\gamma}_0$ is small ($|\bar{\gamma}_0| < 2$), $\begin{pmatrix} 0 & -1 \\ 1 & -\bar{\gamma}_0 \end{pmatrix}$ reduces to a matrix of rotation $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ with $\alpha^2+\beta^2 = 1$.

$$\begin{pmatrix} 0 & -1 \\ 1 & -\bar{\gamma}_0 \end{pmatrix} = Q^{-1}\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} Q$$

with

$$Q = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}, Q^{-1} = \frac{1}{\beta}\begin{pmatrix} \beta & -\alpha \\ 0 & 1 \end{pmatrix}, 2\alpha = -\bar{\gamma}_0.$$

Then,

$$\begin{aligned} e^{t\begin{pmatrix} 0 & -1 \\ 1 & -\bar{\gamma}_0 \end{pmatrix}} &= Q^{-1}e^{t\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}}Q = e^{t\alpha}Q^{-1}e^{t\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}}Q = e^{t\alpha}Q^{-1}\begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix} Q = \\ &= \frac{e^{t\alpha}}{\beta}\begin{pmatrix} \beta & -\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\beta t & \alpha\cos\beta t-\beta\sin\beta t \\ \sin\beta t & \alpha\sin\beta t+\beta\cos\beta t \end{pmatrix} = \\ &= \frac{e^{t\alpha}}{\beta}\begin{pmatrix} \beta\cos\beta t-\alpha\sin\beta t & -\sin\beta t \\ \sin\beta t & \alpha\sin\beta t+\beta\cos\beta t \end{pmatrix} \end{aligned}.$$

Set $\beta = \cos\varphi, \alpha = \sin\varphi$. We find:

$$e^{t\begin{pmatrix} 0 & -1 \\ 1 & -\bar{\gamma}_0 \end{pmatrix}} = \frac{e^{t\alpha}}{\beta}\begin{pmatrix} \cos(\beta t+\varphi) & -\sin\beta t \\ \sin\beta t & \cos(\beta t-\varphi) \end{pmatrix}.$$

Observe that the multiplicative factor tends to 1 as $\bar{\gamma}_0$ tends to zero. $\square$

Let now A be an arbitrary 2 by 2 matrix close to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which reads

$$A = \begin{pmatrix} a & -c^2 \\ c^2 & b \end{pmatrix},$$

has complex eigenvalues and determinant equal to 1.

Lemma 3. *There exists then $\beta_1 = \cos\varphi_1 > 0$ such that*

$$A = \frac{1}{\cos\varphi_1}\begin{pmatrix} \cos(\beta_1 t_1 + \varphi_1) & -\sin\beta_1 t_1 \\ \sin\beta_1 t_1 & \cos(\beta_1 t_1 - \varphi_1) \end{pmatrix} = e^{-t_1 \sin\varphi_1} e^{t_1 \begin{pmatrix} 0 & -1 \\ 1 & 2\sin\varphi_1 \end{pmatrix}}.$$

Proof. We identify

$$a\cos\varphi_1 = \cos(\beta_1 t_1 + \varphi_1) \quad b\cos\varphi_1 = \cos(\beta_1 t_1 - \varphi_1).$$

Thus,

$$\cos(\beta_1 t_1) = \frac{a+b}{2}$$
$$\sin(\beta_1 t_1) = \frac{(b-a)}{2\tan\varphi_1}$$

The first equation can be solved since A has complex eigenvalues and $|trA| < 2$.

The second equation yields then:

$$\tan\varphi_1 = \frac{b-a}{2\sqrt{1-\frac{(a+b)^2}{4}}} = \frac{b-a}{\sqrt{4-(a+b)^2}}$$

We also have

$$c^2 = \frac{\sin\beta_1 t_1}{\cos\varphi_1}$$

i.e.

$$c^2\cos\varphi_1 = \frac{(b-a)\cos\varphi_1}{2\sin\varphi_1}$$
$$\frac{b-a}{2} = c^2\sin\varphi_1.$$

Since $\det A = ab + c^4 = 1, \quad c^2$ is equal to $\sqrt{1-ab}$.

Thus,

$$\sin\varphi_1 = \frac{b-a}{2\sqrt{1-ab}}, \ \cos\ \varphi_1 = \sqrt{1-\frac{(b-a)^2}{4(1-ab)}}$$
$$\tan\varphi_1 = \frac{b-a}{\sqrt{4(1-ab)-(b-a)^2}} = \frac{b-a}{\sqrt{4-(a+b)^2}}.$$

The compatibility follows. ∎

Observation 1 $1 - ab > \frac{(b+a)^2}{4} - ab = \frac{(b-a)^2}{4}$.

Observation 2 $\beta_1 = \cos\varphi_1$ can tend to zero here. Then since $\beta_1 t_1$ can be chosen close to zero ($\cos\beta_1 t_1 = \frac{a+b}{2}$), $\sin\beta_1 t_1$ is of the order of $\beta_1 t_1 = (\cos\varphi_1)t_1$. Thus,

$$(\cos\varphi_1)t_1 \sim \frac{b-a}{2}\frac{\cos\varphi_1}{\sin\varphi_1}.$$

i.e. $t_1 \sim \frac{b-a}{2\sin\varphi_1} = \sqrt{1-ab}$. a and b have both to tend to 1 so that t_1 tends to zero. On the other hand, coming back to $e^{t_1\begin{pmatrix}0 & -1\\ 1 & -\bar{\gamma}_0\end{pmatrix}}$, we have

$$2\alpha_1 = -\bar{\gamma}_0$$

and $|\alpha_1|$ tends to 1 so that $|\bar{\gamma}_0|$ is at most 2.

Observation 3 t_1 is positive and tends to zero as c^2 tends to zero. (This follows readily if β_1 does not tend to zero. If β_1 tends to zero, the claim follows from Observation 2).

Next, we prove

Lemma 4. *Let* $A = \begin{pmatrix} a & -\gamma \\ \delta & b\end{pmatrix}$ *, with* $a+b<2, ab<1, a, b$ *close to 1,* $0<\delta<\gamma, ab+\gamma\delta = 1, \delta(\gamma-\delta)$ *small enough. Then, there exists* $\tilde{a}, \tilde{b}, c$ *with* $\tilde{a}+\tilde{b}<2, \tilde{a}, \tilde{b}$ *close to* $1, \tilde{a}\tilde{b}<1, \tilde{a}\tilde{b}+c^4 = 1$ *and* $\alpha, \beta, \beta_1, \alpha<1, \alpha$ *close to* $1, \alpha^2+\beta^2\beta_1^2 = 1$ *such that*

$$\begin{pmatrix} a & -\gamma \\ \delta & b\end{pmatrix} = \begin{pmatrix} \tilde{a} & -c^2 \\ c^2 & \tilde{b}\end{pmatrix}\begin{pmatrix} \alpha & -\beta_1^2 \\ \beta^2 & \alpha\end{pmatrix}.$$

Proof. We have

$$\begin{matrix} a = \tilde{a}\alpha - c^2\beta^2 & b = \tilde{b}\alpha - c^2\beta_1^2 \\ \delta = c^2\alpha + \tilde{b}\beta^2 & \gamma = \tilde{a}\beta_1^2 + c^2\alpha \end{matrix} \qquad \alpha^2+\beta^2\beta_1^2 = 1\,.$$

We may replace the condition $\delta = c^2\alpha + \tilde{b}\beta^2$ with $\tilde{a}\tilde{b}+c^4 = 1$.

We then have

$$\begin{matrix} \tilde{b} = \frac{b+c^2\beta_1^2}{\alpha}, & \tilde{a}\beta_1^2 = \gamma - c^2\alpha \\ a = \tilde{a}\alpha - c^2\frac{(1-\alpha^2)}{\beta_1^2} & \tilde{a}\tilde{b}+c^4 = 1 \end{matrix} \qquad \alpha^2+\beta^2\beta_1^2 = 1$$

which rereads

$$\beta_1^2 = \frac{\alpha(\gamma-c^2\alpha) - c^2(1-\alpha^2)}{a} = \frac{\alpha\gamma - c^2}{a}$$

$$\tilde{b} = \frac{b+c^2\beta_1^2}{\alpha}$$

$$\tilde{a} = \frac{a(\gamma-c^2\alpha)}{\alpha\gamma-c^2} \qquad \alpha^2+\beta^2\beta_1^2 = 1$$

$$(b+c^2\beta_1^2)(\gamma-c^2\alpha) = \alpha\beta_1^2(1-c^4).$$

The last equation yields

$$\beta_1^2(\alpha - \gamma c^2) = b(\gamma - c^2\alpha). \tag{***}$$

Combining this equation with the first equation, we find

$$(\alpha - \gamma c^2)(\alpha\gamma - c^2) = ab(\gamma - c^2\alpha). \tag{****}$$

This equation ties α and c.

If we can solve (****) with c small, α close to $1 (\alpha^2 < 1)$, then (***) gives us $\beta_1^2 > 0$ (provided $c^2 < \gamma$). We can then find $\tilde{a}$ and $\tilde{b}$. As α tends to 1, with $\gamma - c^2$ bounded away from zero, $\frac{\tilde{a}}{a}$ tends to 1. If γc^2 is small enough, $\tilde{b}$ will be very close to b. β can be computed from

$$\beta^2 = \frac{1-\alpha^2}{\beta_1^2}$$

(****) rereads

$$\gamma c^4 - c^2(\alpha\gamma^2 + \alpha - ab\alpha) + \gamma(\alpha^2 - ab) = 0 \quad \text{i.e.}$$
$$c^4 - c^2\alpha(\gamma + \delta) + (\alpha^2 - 1 + \gamma\delta) = 0.$$

The discriminant is

$$(\alpha\gamma^2 + \alpha - ab\alpha)^2 - 4\gamma^2(\alpha^2 - ab) \geq \alpha^2((\gamma^2 + 1 - ab)^2 - 4\gamma^2(1 - ab)).$$

Observe that

$$(\gamma^2 + 1 - ab)^2 - 4\gamma^2(1 - ab) = (\gamma^2 + \gamma\delta)^2 - 4\gamma^2 \cdot \gamma\delta = (\gamma^2 - \gamma\delta)^2 = \gamma^2(\gamma - \delta)^2 > 0.$$

Hence (****) has two positive roots as $|\alpha|$ tends to $1 (ab < 1)$. For $\alpha = 1$, the equation becomes

$$c^4 - c^2(\gamma + \delta) + \delta\gamma = 0.$$

The two solutions are

$$c^2 = \gamma \text{ and } c^2 = \delta.$$

Assume $\delta < \gamma$, we choose $c^2 = \delta$ and derive from (****) that

$$\beta_1^2 = \frac{b(\gamma - \delta)}{1 - \gamma\delta} = \frac{\gamma - \delta}{a} > 0.$$

Furthermore, since $\beta^2 = 0$,

$$\tilde{a} = a, \tilde{b} = b + \frac{\delta(\gamma - \delta)}{a}.$$

Thus, if $\delta(\gamma-\delta)$ is small enough, $\tilde{a}$ and $\tilde{b}$ are close to a, b and satisfy our requirements.

If $\alpha < 1$ is very close to 1, all those arguments proceed with a solution c^2 as close as we may wish to δ and $\beta^2 = \frac{1-\alpha^2}{\beta_1^2}$ close to zero.

We thus need $\gamma > \delta$ in order to solve our equation. $\gamma - \delta$ can be as close as we wish to zero, α will be taken closer to 1.

We consider now the case $\delta > \gamma$. We observe that

$$\begin{pmatrix} a & -\gamma \\ \delta & b \end{pmatrix} = \begin{pmatrix} a & \delta \\ -\gamma & b \end{pmatrix}^t = \begin{pmatrix} a & \bar{\gamma} \\ -\bar{\delta} & b \end{pmatrix}^t \text{ with } \bar{\delta} < \bar{\gamma}.$$

We solve

$$\begin{pmatrix} a & \bar{\gamma} \\ -\bar{\delta} & b \end{pmatrix} = \begin{pmatrix} \tilde{a} & c^2 \\ -c^2 & \tilde{b} \end{pmatrix} \begin{pmatrix} \alpha & \beta_1^2 \\ -\beta^2 & \alpha \end{pmatrix}.$$

This yields

$$a = \tilde{a}\alpha - c^2\beta^2, b = \tilde{b}\alpha - \beta_1^2 c^2, \bar{\gamma} = \tilde{a}\beta_1^2 + c^2\alpha, \bar{\delta} = c^2\alpha + \tilde{b}\beta^2$$

exactly as above, with $\bar{\delta} < \bar{\gamma}$.

Thus, this equation may be solved and consequently, we may write

Lemma 5.

$$\begin{pmatrix} a & -\gamma \\ \delta & b \end{pmatrix} = \begin{pmatrix} \alpha & -\beta^2 \\ \beta_1^2 & \alpha \end{pmatrix} \begin{pmatrix} \tilde{a} & -c^2 \\ c^2 & \tilde{b} \end{pmatrix} \textit{ for } \delta > \gamma.$$

We want to show how to generate the matrices

$$\begin{pmatrix} \alpha & -\beta_1^2 \\ \beta^2 & \alpha \end{pmatrix} \quad \alpha^2 + \beta^2\beta_1^2 = 1$$

with β_1 and β close to zero, $\beta_1^2 > \beta^2$ or vice-versa.

We compute the product of two matrices A, A_1

$$A = \begin{pmatrix} a & -\sqrt{1-ab} \\ \sqrt{1-ab} & b \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} a_1 & -\sqrt{1-a_1b_1} \\ \sqrt{1-a_1b_1} & b_1 \end{pmatrix}.$$

We find

$$AA_1 = \begin{pmatrix} aa_1 - \sqrt{1-ab}\sqrt{1-a_1b_1} & -a\sqrt{1-a_1b_1} - b_1\sqrt{1-ab} \\ a_1\sqrt{1-ab} + b\sqrt{1-a_1b_1} & bb_1 - \sqrt{1-ab}\sqrt{1-a_1b_1} \end{pmatrix}.$$

If $a = b_1$ and $b = a_1$, we find

$$AA_1 = \begin{pmatrix} 2ab-1 & -2a\sqrt{1-ab} \\ 2b\sqrt{1-ab} & 2ab-1 \end{pmatrix}.$$

Clearly if $aa_1 = bb_1$ then

$$AA_1 = \begin{pmatrix} \alpha & -\beta_1^2 \\ \beta^2 & \alpha \end{pmatrix}.$$

with $\alpha^2 + \beta^2\beta_1^2 = 1$.

Furthermore, if a is close to b_1 and b is close to a_1, then α is close to $2ab - 1$, which is close to 1 if ab is close to 1. We thus need to worry about

$$\beta_1^2 > \beta^2 \text{ or } \beta^2 > \beta_1^2.$$

We then observe that, since $aa_1 = bb_1$,

$$\beta_1^2 = a\sqrt{1 - a_1b_1} + b_1\sqrt{1 - ab} = a\sqrt{1 - a_1b_1} + \frac{aa_1}{b}\sqrt{1 - ab} =$$

$$= \frac{a}{b}\left(a_1\sqrt{1 - ab} + b\sqrt{1 - a_1b_1}\right) = \frac{a}{b}\beta^2.$$

Taking $a > b$ or $b > a$, we achieve the two occurrences.

Consider now a matrix

$$\bar{A} = \begin{pmatrix} \bar{a} & -\bar{c}^2 \\ \bar{c}^2 & \bar{b} \end{pmatrix} \text{ with } \begin{matrix} \bar{a}\bar{b} + \bar{c}^4 = 1 \\ 0 < \bar{a} + \bar{b} < 2 \end{matrix}.$$

$\bar{a}, \bar{b}$ close to 1, fixed.

Consider a small angle $t_2 > 0$ and the product

$$A_{t_2} = \begin{pmatrix} \bar{a} & -\bar{c}^2 \\ \bar{c}^2 & \bar{b} \end{pmatrix} \begin{pmatrix} \cos t_2 & \sin t_2 \\ -\sin t_2 & \cos t_2 \end{pmatrix} = \begin{pmatrix} \bar{a}\cos t_2 + \bar{c}^2 \sin t_2 & \bar{a}\sin t_2 - \bar{c}^2\cos t_2 \\ \bar{c}^2\cos t_2 - \bar{b}\sin t_2 & \bar{b}\cos t_2 + \bar{c}^2\sin t_2 \end{pmatrix}.$$

As t_2 tends to zero, this matrix gets closer and closer to $\bar{A}$ and assumes the form

$$\begin{pmatrix} a_0 & -\gamma \\ \delta & b_0 \end{pmatrix}.$$

Assume that

$$\bar{b} > \bar{a} \text{ i.e } \delta < \gamma. \tag{1}$$

The other case is similar.

We apply then Lemma 4 and write

$$A_{t_2} = \begin{pmatrix} \tilde{a} & -c^2 \\ c^2 & \tilde{b} \end{pmatrix} \begin{pmatrix} \alpha & -\beta_1^2 \\ \beta^2 & \alpha \end{pmatrix}$$

c solves

$$c^4 - c^2\alpha(\gamma+\delta) + (\alpha^2 - 1 + \gamma\delta) = 0.$$

Assuming that

$$1 - \gamma\delta < \alpha^2 < 1, \gamma + \delta = 2\bar{c}^2\cos t_2 - (\bar{a} + \bar{b})\sin t_2 > 0. \tag{2}$$

We find

$$c^2 = \frac{\alpha(\gamma+\delta) - \sqrt{\alpha^2(\gamma+\delta)^2 - 4(\alpha^2 - 1 + \gamma\delta)}}{2} = \frac{\alpha(\gamma+\delta) - \sqrt{\alpha^2(\gamma-\delta)^2 + 4(1-\alpha^2)(1-\gamma\delta)}}{2}.$$

We then have

$$a_0\beta_1^2 = \alpha\gamma - c^2 = \frac{\alpha(\gamma-\delta) + \sqrt{\alpha^2(\gamma-\delta)^2 + 4(1-\alpha^2)(1-\gamma\delta)}}{2}. \tag{3}$$

The positivity is warranted by $\bar{b} > \bar{a}$.

Observe that

$$|\delta(\gamma - \delta)| \leq |\bar{a} - \bar{b}||\sin t_2|.$$

For fixed $\bar{a}$ and $\bar{b}$, this can be made as small as we wish by taking t_2 to be small. Observe also that, as we reduce $\begin{pmatrix} a & -\gamma \\ \delta & b \end{pmatrix}$ into $\begin{pmatrix} \tilde{a} & -c^2 \\ c^2 & \tilde{b} \end{pmatrix}\begin{pmatrix} \alpha & -\beta_1^2 \\ \beta^2 & \alpha \end{pmatrix}$, $\tilde{a} = a\frac{(\gamma - c^2\alpha)}{\alpha\gamma - c^2} = \frac{a}{\alpha} + \frac{ac^2}{\alpha}\frac{1-\alpha^2}{\alpha\gamma - c^2} = \frac{a}{\alpha} + \frac{ac^2}{\alpha}(1-\alpha^2)O\left(\frac{1}{\sqrt{1-\alpha^2}}\right) = \frac{a}{\alpha} + O(\sqrt{1-\alpha^2})$. Thus, if we can choose α very close to 1, for $\bar{a}$ and $\bar{b}$ fixed with $\bar{a}\bar{b} < 1, \bar{a} + \bar{b} < 2$, then $\tilde{a}\tilde{b} < 1, \tilde{a} + \tilde{b} < 2(\tilde{a}, \tilde{b}$ very close to $\bar{a}, \bar{b})$. Indeed, as α tends to 1, $\tilde{a}$ tends to a, c^2 tends to δ and $c^2\beta_1^2$ is $O(\delta(\gamma-\delta) + \delta\sqrt{1-\alpha^2})$. This tends to zero and α tends to 1.

We then have

Lemma 6. *If $\bar{a}, \bar{b}$ are chosen appropriately, there exists $t_2 > 0$ small such that the matrix $\begin{pmatrix} \alpha & -\beta_1^2 \\ \beta^2 & \alpha \end{pmatrix}$ can be written as*

$$\begin{pmatrix} a & -\sqrt{1-ab} \\ \sqrt{1-ab} & b \end{pmatrix}\begin{pmatrix} b & -\sqrt{1-ab} \\ \sqrt{1-ab} & a \end{pmatrix} \text{ with } 0 < a + b < 2.$$

Observation In the reduction of Lemma 4, 5, α is a free parameter close enough to 1. β_1^2 depends on $\alpha, \bar{a}, \bar{b}, t_2$. Lemma 6 states that we can find $\bar{a}, \bar{b}$ **and** α, also a and b so that the equation $A_{t_2} = \begin{pmatrix} \tilde{a} & -c^2 \\ c^2 & \tilde{b} \end{pmatrix}\begin{pmatrix} a & -\sqrt{1-ab} \\ \sqrt{1-ab} & b \end{pmatrix}\begin{pmatrix} b & -\sqrt{1-ab} \\ \sqrt{1-ab} & a \end{pmatrix}$ is solvable in t_2.

Proof. We then should have

$$\alpha = 2ab - 1 \tag{4}$$

$$\beta_1^2 = 2a\sqrt{1-ab} \tag{5}$$

(2) becomes

$$4ab(1-ab) < \gamma\delta, ab < 1, \gamma+\delta > 0 \tag{6}$$

and (3) becomes

$$4aa_0\sqrt{1-ab} = (2ab-1)(\gamma-\delta) + \sqrt{(2ab-1)^2(\gamma-\delta)^2 + 16(1-\gamma\delta)ab(1-ab)}. \tag{7}$$

Assume that

$$4aa_0\sqrt{1-ab} > (2ab-1)(\gamma-\delta). \tag{8}$$

Then, (7) yields:

$$16a^2a_0^2\sqrt{1-ab} + \frac{(2ab-1)^2(\gamma-\delta)^2}{\sqrt{1-ab}} - 8aa_0(2ab-1)(\gamma-\delta) = \tag{9}$$

$$= \frac{(2ab-1)^2(\gamma-\delta)^2}{\sqrt{1-ab}} + 16(1-\gamma\delta)ab\sqrt{1-ab}$$

i.e.

$$2aa_0^2\sqrt{1-ab} - a_0(2ab-1)(\gamma-\delta) = 2(1-\gamma\delta)b\sqrt{1-ab}. \tag{10}$$

Thus,

$$2\sqrt{1-ab}(aa_0^2 - b(1-\gamma\delta)) = a_0(2ab-1)(\gamma-\delta). \tag{11}$$

Observe that

$$a_0 = \bar{a}\cos t_2 + \bar{c}^2\sin t_2, \gamma = \bar{c}^2\cos t_2 - \bar{a}\sin t_2, \delta = \bar{c}^2\cos t_2 - \bar{b}\sin t_2 \tag{12}$$

t_2 should be positive and small.
Replacing in (11) and using $\bar{a}\bar{b} = 1-\bar{c}^4$, we derive

$$2\sqrt{1-ab}\bar{a}(a\bar{a}-b\bar{b}) = (\bar{a}\cos t_2 + \bar{c}^2\sin t_2)(2ab-1)(\bar{b}-\bar{a})\sin t_2 + \tag{13}$$

$$+2\sin t_2\sqrt{1-ab}\left(\sin t_2(\bar{c}^4(b-a) + \bar{a}(a\bar{a}-b\bar{b})) + \cos t_2\bar{c}^2(a(b-a)+b\bar{b}-a\bar{a})\right).$$

Observe that

$$b - a = \frac{a}{\bar{b}}(\bar{a} - \bar{b}) + \frac{b\bar{b} - a\bar{a}}{\bar{b}}. \tag{14}$$

Thus,

$$2\sqrt{1-ab}\bar{a}(a\bar{a} - b\bar{b}) = (\bar{b} - \bar{a})\sin t_2 \big((\bar{a}\cos t_2 + \bar{c}^2 \sin t_2)(2ab-1) - \tag{15}$$
$$-2\frac{a}{\bar{b}}\sqrt{1-ab}\bar{c}^4 \sin t_2 - 2\frac{a\bar{a}}{\bar{b}}\sqrt{1-ab}\bar{c}^2\cos t_2\big) + 2(a\bar{a} - b\bar{b})\sqrt{1-ab} \times \sin t_2 \times$$
$$\times \left(\bar{a}\sin t_2 - \sin t_2 \frac{\bar{c}^4}{\bar{b}} - \cos t_2 \bar{c}^2\left(\frac{\bar{a}}{\bar{b}} + 1\right)\right).$$

Assume now that

$$\bar{b} > \bar{a}, a\bar{a} > b\bar{b}, a\bar{a} - b\bar{b} = O((\bar{b} - a)^2). \tag{16}$$

$$ab < 1, a + b < 2, a, b \text{ close to } 1. \tag{17}$$

Since $1 - ab$ is small and $2ab - 1$ is close to 1 while $a\bar{a} - b\bar{b} > 0, a\bar{a} - b\bar{b} = O((\bar{b} - \bar{a})^2)$, we can solve (15) by implicit function theorem and find $t_2 > 0$ small. Indeed (15) rewrites under (16):

$$2\sqrt{1-ab}\bar{a}\frac{(a\bar{a} - b\bar{b})}{\bar{b} - \bar{a}} = \sin t_2(\bar{a}(2ab-1) + o(1)). \tag{18}$$

We need therefore to fulfill (16) and (17), also (8).

Consider

$$1 > \bar{b} > \bar{a}. \tag{19}$$

Let $\bar{c}$ be such that $\bar{c}^4 = 1 - \bar{a}\bar{b}$.

Take

$$a = \frac{b\bar{b}}{\bar{a}} + \varepsilon, \varepsilon > 0 \text{ tending to zero }, b < 1. \tag{20}$$

For ε small enough, if $\frac{\bar{b}}{\bar{a}}$ is close enough to 1 (in function of b)

$$ab = \frac{b^2\bar{b}}{\bar{a}} + \varepsilon b < 1. \tag{21}$$

Also

$$a+b=\frac{b\bar{b}}{\bar{a}}+\varepsilon+b=b\left(\frac{\bar{b}}{\bar{a}}+1\right)+\varepsilon<2. \tag{22}$$

We then need

$$a\bar{a}-b\bar{b}=O((\bar{b}-\bar{a})^2) \tag{23}$$

i.e.

$$\varepsilon=O((\bar{b}-\bar{a})^2)=O((\frac{\bar{b}}{\bar{a}}-1)^2)$$

and this is easy to satisfy.

The proximity of a and b to 1 depends only on the proximity of b and $\frac{\bar{b}}{\bar{a}}$ to 1.

Finally, we need (6) i.e.

$$4ab(1-ab)<(\bar{c}^2\cos t_2-\bar{a}\sin t_2)(\bar{c}^2\cos t_2-\bar{b}\sin t_2) \tag{24}$$

$$2\bar{c}^2\cos t_2-(\bar{b}+\bar{a})\sin t_2>0 \tag{25}$$

(25) follows from the fact that t_2 is small.

For (24), we observe that as $\frac{\bar{b}}{\bar{a}}$ tends to 1 and $1-ab$ to zero, $\bar{a}\bar{b}$ can be kept away from 1 so that $\bar{c}^2$ is far from zero and t_2 tends to zero. (24) follows.

We also assumed (8) i.e.

$$(2ab-1)(\bar{b}-\bar{a})\sin t_2<4a(\bar{a}\cos t_2+\bar{c}^2\sin t_2)\sqrt{1-ab}. \tag{25$'$}$$

Using (18), this rereads:

$$2\sqrt{1-ab}\,(a\bar{a}-b\bar{b})(1+o(1))<4a\bar{a}\sqrt{1-ab}(1+o(1)) \tag{26}$$

which follows readily. □

We now build γ:

We pick up a small interval J and we consider the v-transport over J which is given in Span $\left(\frac{\tilde{\partial}}{\partial x},\frac{\tilde{\partial}}{\partial y}\right)$ by

$$e^{\bar{t}\begin{pmatrix}0 & -1\\ 1 & \frac{-2\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}}\end{pmatrix}}.$$

Assume $-\bar{\gamma} > 0$ for example; $\bar{\gamma}$ will be assumed - there is, after a simple argument, no restriction in this assumption - to be as small as we please. Use Lemma 2 to read this matrix up to a multiplicative factor under the form

$$\begin{pmatrix} \bar{a} & -\bar{c}^2 \\ \bar{c}^2 & \bar{b} \end{pmatrix} \textit{ with } \bar{a}+\bar{b} < 2, \bar{a}, \bar{b} \text{ close to } 1, \bar{a}\bar{b}+\bar{c}^4 = 1.$$

Choose $\beta = \cos\varphi > 0$. Since $-\bar{\gamma} > 0, \alpha = \sin\varphi$ is positive and since $\bar{t}$ is small,

$$\bar{b} = \frac{\cos(\beta\bar{t}-\varphi)}{\cos\varphi} > \frac{\cos(\beta\bar{t}+\varphi)}{\cos\varphi} = \bar{a}.$$

Also

$$\bar{a}\bar{b} = \frac{1}{\cos^2\varphi}(\cos^2\beta\bar{t}\cos^2\varphi - \sin^2\beta\bar{t}\sin^2\varphi) < 1 \tag{27}$$

$$\frac{\bar{b}}{\bar{a}} \text{ tends to 1 as } \bar{\gamma} \text{ tends to zero since } \sin\varphi = \frac{|\bar{\gamma}|}{2}.$$

Pick up $b < 1$. Set $a = \frac{b\bar{b}}{\bar{a}} + \left(\frac{\bar{b}}{\bar{a}} - 1\right)^3$. Adjust $|\bar{\gamma}|$ so small that $\frac{\bar{b}}{\bar{a}} - 1$ is as small as we please with $\bar{b}, \bar{a}$ away from 1 ($\bar{t}$ is given, small, positive). $a, b, \bar{a}\bar{b}$ satisfy (16) and (17). (6) and (8) follow if t_2 is small positive, see (24), (25), (25′), (26).

Using Lemma 6, we rewrite

$$\begin{pmatrix} \bar{a} & -\bar{c}^2 \\ \bar{c}^2 & \bar{b} \end{pmatrix} = \begin{pmatrix} \tilde{a} & -c^2 \\ c^2 & \tilde{b} \end{pmatrix} \begin{pmatrix} a & -\sqrt{1-ab} \\ \sqrt{1-ab} & b \end{pmatrix} \begin{pmatrix} b & -\sqrt{1-ab} \\ \sqrt{1-ab} & a \end{pmatrix} \begin{pmatrix} \cos t_2 & -\sin t_2 \\ \sin t_2 & \cos t_2 \end{pmatrix}$$

with t_2 positive small, $0 < \tilde{a}+\tilde{b} < 2, 0 < a+b < 2, \tilde{a}\tilde{b} < 1, ab < 1$.

Using then Lemma 3, we may write

$$\begin{pmatrix} \bar{a} & -\bar{c}^2 \\ \bar{c}^2 & \bar{b} \end{pmatrix} = \theta e^{t_5\begin{pmatrix} 0 & -1 \\ 1 & 2\sin\varphi_1 \end{pmatrix}} e^{t_4\begin{pmatrix} 0 & -1 \\ 1 & 2\sin\varphi_2 \end{pmatrix}} e^{t_3\begin{pmatrix} 0 & -1 \\ 1 & 2\sin\varphi_3 \end{pmatrix}} e^{t_2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

with $\theta, t_2, t_3, t_4, t_5 > 0$.

All multiplicative factors tend to 1 as t_1 tends to zero. $\int|\gamma'|$ is clearly bounded. $\square$

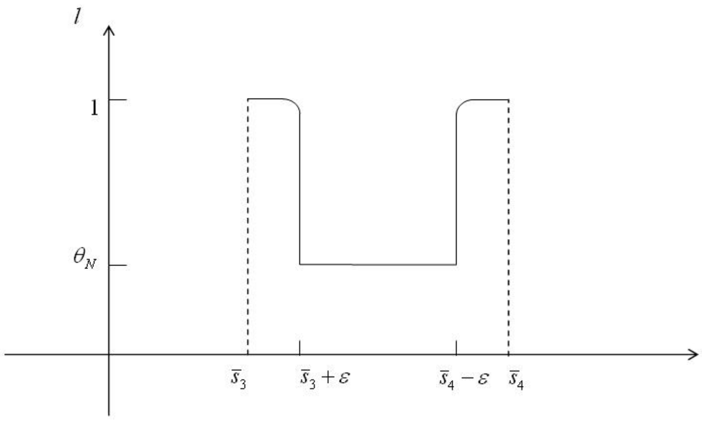
Modification of α into α_N.

We focus on the interval I_0 and we pick up $\varepsilon > 0$ and N large, with $\varepsilon = o\left(\frac{|I_0|}{N}\right)$. We pick up a real

$$\frac{|I_0|}{2\pi(N+1)} < \theta_N < \frac{|I_0|}{2\pi N}$$

in a way which will become clear in a moment.

We build a fucntion ℓ on I_0 as follows:



θ_N is chosen so that $\int_{\bar{s}_3}^{\bar{s}_4} \frac{ds_1}{\ell(s_1)} = 2\pi N + |I_0|$

Setting

$$s_2 = \int_{\bar{s}_3}^{s_1} \frac{d\tau}{\ell(\tau)} \qquad \text{i.e.}$$

$$ds_2 = \frac{ds_1}{\ell(s_1)}, \frac{\partial}{\partial s_2} = \ell(s_1)\frac{\partial}{\partial s_1}$$

we consider the differential equation

$$\ell(s_1)\frac{\partial}{\partial s_1}\left(\ell(s_1)\frac{\partial}{\partial s_1}\right)u + u = 0 \quad \text{ on } I_0$$

which rereads

$$\frac{\partial^2 u}{\partial s_2^2} + u = 0 \text{ on an interval of length } 2\pi N + |I_0|$$

starting at $\bar{s}_3$ i.e. on $[\bar{s}_3, \bar{s}_4 + 2\pi N]$.

Thus, the solutions of

$$[v,[v,\xi]] = -\xi \text{ on } I_0$$

and of

$$[\ell v,[\ell v,\xi]] = -\xi \text{ on } I_0$$

with the same initial data $\xi, [v,\xi]$ match at $\bar{s}_4$.

Computation of ξ_N.

We compute ξ_N.

ξ_N satisfies the differential equation:

$$[\varphi v, [\varphi v, \xi_N]] = -\xi_N \quad \varphi = \ell.$$

We know that $\xi_N(\bar{s}_3) = \bar{\xi}$.

We need to compute $[-\varphi v, \xi_N](\bar{s}_3) = [-v, \xi_N](\bar{s}_3)$.

Lemma 7. $-[v, \xi_N](\bar{s}_3) = -2\frac{(x-\bar{\gamma}y)(0)}{\sqrt{1+\bar{\gamma}^2}(x^2+y^2)(0)} v + \sqrt{1+\bar{\gamma}^2}\frac{\partial}{\partial y}$.

Proof. ξ satisfies

$$[-v, [-v, \xi]] = \xi + \gamma[\xi, v] - (\xi \cdot \gamma)v$$

and γ incurs a jump at $\bar{s}_3$ from $\frac{-2\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}} = -2\bar{\gamma}_1$ to 0. γ is a function of s, which is a function of τ, the time along $-v$. We are here taking $s = x^2 + y^2$ and thinking of $\gamma' ds(\xi)$ as $d\gamma(\xi)$. We have

$$\frac{d\gamma}{d\tau} = +\frac{d\gamma}{ds} \cdot \frac{ds}{d\tau}$$

so that

$$\gamma' = +\frac{d\gamma}{d\tau} \cdot \frac{d\tau}{ds} = \frac{d\gamma}{d\tau} \times \frac{1}{2\bar{\gamma}(x^2+y^2)}.$$

Thus,

$$[-v, \xi](\bar{s}_3^+) - [-v, \xi](\bar{s}_3^-) = -\frac{2\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}} \times \frac{1}{2\bar{\gamma}(x^2+y^2)} ds(\xi)(\bar{s}_3)v =$$
$$= -\frac{2(x-\bar{\gamma}y)(0)}{\sqrt{1+\bar{\gamma}^2}(x^2+y^2)(0)} \cdot v.$$

On the other hand

$$[-v, \xi](\bar{s}_3^-) = \left[-20\frac{\partial}{\partial\theta} + (y+\bar{\gamma}x)\frac{\partial}{\partial x} - (x-\bar{\gamma}y)\frac{\partial}{\partial y}, \frac{\partial}{\partial x} - \bar{\gamma}\frac{\partial}{\partial y}\right] \cdot \frac{1}{\sqrt{1+\bar{\gamma}^2}} = \sqrt{1+\bar{\gamma}^2}\frac{\partial}{\partial y}.$$

Thus, ξ_N satisfies

$$\left\{\begin{array}{l} [-\varphi v, [-\varphi v, \xi_N]] = -\xi_N \\ \xi_N(\bar{s}_3) = \bar{\xi} \\ [-\varphi v, \xi_N](\bar{s}_3) = \sqrt{1+\bar{\gamma}^2}\frac{\partial}{\partial y} - 2\frac{(x-\bar{\gamma}y)(0)}{\sqrt{1+\bar{\gamma}^2}(x^2+y^2)(0)} v. \end{array}\right. \tag{28}$$

Let

$$e_1 = 20\frac{\partial}{\partial\theta}, e_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, e_3 = \varphi v$$

φ is a function of $s = x^2 + y^2$. Thus,

$$ds(e_1) = ds(e_2) = 0$$

and

$$[e_1, v] = [e_2, v] = [e_1, \varphi v] = [e_2, \varphi v] = 0$$

since

$$\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right] = 0.$$

Observe that

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = -\frac{v - 20\frac{\partial}{\partial\theta} + (y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})}{\bar{\gamma}} \cdot \sqrt{1+\bar{\gamma}^2} = \frac{e_1 - e_2 - \varphi v}{\bar{\gamma}}\sqrt{1+\bar{\gamma}^2}.$$

Observe also that

$$\frac{\partial}{\partial x} = \frac{x(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) + y(\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})}{x^2+y^2} = \frac{x}{x^2+y^2}\sqrt{1+\bar{\gamma}^2}\frac{e_1 - e_2 - \varphi v}{\bar{\gamma}} + \frac{y}{x^2+y^2}e_2$$

$$\frac{\partial}{\partial y} = \frac{y}{x^2+y^2} \cdot \frac{e_1 - e_2 - \varphi v}{\bar{\gamma}} \cdot \sqrt{1+\bar{\gamma}^2} - \frac{xe_2}{x^2+y^2}.$$

Thus,

$$\begin{cases} \bar{\xi} = \frac{\partial}{\partial x} - \bar{\gamma}\frac{\partial}{\partial y} = \frac{x-\bar{\gamma}^y}{x^2+y^2}\sqrt{1+\bar{\gamma}^2}\frac{(e_1-e_2-\varphi v)}{\bar{\gamma}} + \frac{(y+\bar{\gamma}x)e_2}{x^2+y^2} \\ [-\varphi v, \xi_N](\bar{s}_3) = \sqrt{1+\bar{\gamma}^2}\frac{y}{x^2+y^2}\frac{(e_1-e_2-\varphi v)}{\bar{\gamma}} - \frac{\sqrt{1+\bar{\gamma}^2}xe_2}{x^2+y^2} - 2\frac{(x-\bar{\gamma}y)}{x^2+y^2}\frac{\varphi v}{\sqrt{1+\bar{\gamma}^2}} \end{cases}.$$

Here, x, y are taken at s_3 and we will label them $x(0), y(0)$.

We solve then (28). We derive:

Proposition 2. *Let s_2 be the time parameter along $-\varphi v$*

$$\xi_N = \left(\frac{(x-\bar{\gamma}y)(0)}{(x^2+y^2)(0)}\frac{1}{\sqrt{1+\bar{\gamma}^2}}\left(\frac{e_1 - e_2 - \varphi v}{\bar{\gamma}}\right) + \frac{(y+\bar{\gamma}x)(0)}{(x^2+y^2)(0)}e_2\right)\cos_2 +$$

$$+\left(\sqrt{1+\bar{\gamma}^2}\frac{y(0)}{(x^2+y^2)(0)}\frac{e_1 - e_2 - \varphi v}{\bar{\gamma}} - \frac{\sqrt{1+\bar{\gamma}^2}x(0)e_2}{(x^2+y^2)(0)} - 2\frac{(x-\bar{\gamma}y)(0)}{(x^2+y^2)(0)}\frac{\varphi v}{\sqrt{1+\bar{\gamma}^2}}\right)\sin s_2.$$

Corollary 2. *Given a fixed connection on M, $\xi_N\cdot\xi_N, \xi_N\cdot[\varphi v,\xi_N], [\varphi v,\xi_N]\cdot\xi_N$ and $[\varphi v,\xi_N]\cdot[\varphi v,\xi_N]$ are bounded independently on M transversally to v.*

Proof. Observe that $ds_2(\varphi v) = -1$ and $ds_2(e_1) = ds_2(e_2) = 0$. The only unbounded terms could come from a derivative taken on φ. But φ is multiplied by v. Hence the claim.

Observe that

$$[\varphi v, e_1] = [\varphi v, e_2] = 0.$$

Thus, denoting γ_{s_2} the one-parameter group of $-\varphi v$

$$D\gamma_{s_2}(e_1) = e_1 \quad D\gamma_{s_2}(e_2) = e_2$$

$(x,y,\theta)(s_2)$ is derived from $(x,y,\theta)(0)$ through the use of γ_{s_2}.

We can use this fact and reread the differential equation

$$\overline{\dot{(x,y,\theta)}} = \xi_N(x,y,\theta).$$

Indeed, we then have

$$D\gamma_{s_2}\left(\overline{\dot{(x,y,\theta)(0)}}\right) - \dot{s}_2\varphi v =$$

$$A\varphi v + \cos_2\left(\frac{(x-\bar{\gamma}y)(0)}{(x^2+y^2)(0)} \times \frac{1}{\sqrt{1+\bar{\gamma}^2}} D\gamma_{s_2}\left(\frac{e_1-e_2}{\bar{\gamma}}\right) + \frac{(y+\bar{\gamma}x)(0)}{(x^2+y^2)(0)} D\gamma_{s_2}(e_2)\right) +$$

$$+\left(\sqrt{1+\bar{\gamma}^2}\frac{y}{x^2+y^2}(0) D\gamma_{s_2}\left(\frac{e_1-e_2}{\bar{\gamma}}\right) - \frac{\sqrt{1+\bar{\gamma}^2}x(0)}{(x^2+y^2)(0)} D\gamma_{s_2}(e_2)\right)\sin s_2.$$

Set

$$x(0) = \rho\cos\psi$$
$$y(0) = \rho\sin\psi.$$

Then,

$$e_2 = y(0)\frac{\partial}{\partial x} - x(0)\frac{\partial}{\partial y} = -\rho\frac{\partial}{\partial\psi}.$$

and

$$\overline{\dot{(x,y,\theta)(0)}} = \frac{\cos s_2}{\rho}\left(\frac{\cos\psi - \bar{\gamma}\sin\psi}{\bar{\gamma}\sqrt{1+\bar{\gamma}^2}}\left(20\frac{\partial}{\partial\theta} + \rho\frac{\partial}{\partial\psi}\right) - (\sin\psi + \bar{\gamma}\cos\psi)\rho\frac{\partial}{\partial\psi}\right) +$$

$$+\frac{\sin s_2}{\rho}\left(\sqrt{1+\bar{\gamma}^2}\frac{\sin\psi}{\bar{\gamma}}\left(20\frac{\partial}{\partial\theta} + \rho\frac{\partial}{\partial\psi}\right) + \sqrt{1+\bar{\gamma}^2}\cos\psi\rho\frac{\partial}{\partial\psi}\right).$$

Observe now that

$$\overline{\dot{(x,y,\theta)(0)}} = \dot{\theta}\frac{\partial}{\partial\theta} + \rho\dot{\psi}\frac{\partial}{\partial\psi}$$

since $\rho^2 = (x^2+y^2)(0)$ is a constant.

We thus derive

Proposition 3. *The differential equation* $\dot{\overline{(x,y,\theta)}} = \xi_N(x,y,\theta)$ *rereads*

$$\rho\dot{\psi} = \cos s_2\left(\frac{\cos\psi - \bar{\gamma}\sin\psi}{\bar{\gamma}\sqrt{1+\bar{\gamma}^2}} - (\sin\psi + \bar{\gamma}\cos\psi)\right) + \sin s_2\left(\sqrt{1+\bar{\gamma}^2}\frac{\sin\psi}{\bar{\gamma}} + \sqrt{1+\bar{\gamma}^2}\cos\psi\right)$$
$$\rho\dot{\theta} = 20\left(\cos s_2\cdot\frac{\cos\psi - \bar{\gamma}\sin\psi}{\bar{\gamma}\sqrt{1+\bar{\gamma}^2}} + \frac{\sin s_2\sqrt{1+\bar{\gamma}^2}\sin\psi}{\bar{\gamma}}\right)$$
$$\rho\dot{s}_2 = \cos s_2\left(\frac{\cos\psi - \bar{\gamma}\sin\psi}{\bar{\gamma}\sqrt{1+\bar{\gamma}^2}}\right) + \left(\sqrt{1+\bar{\gamma}^2}\frac{\sin\psi}{\bar{\gamma}} + 2\frac{\cos\psi - \bar{\gamma}\sin\psi}{\sqrt{1+\bar{\gamma}^2}}\right)\sin s_2.$$

If we set $\frac{\partial}{\partial\tau} = \rho\bar{\gamma}\frac{\partial}{\partial t}$, this becomes

$$\frac{\partial\psi}{\partial\tau} = \cos s_2\left(\frac{\cos\psi - \bar{\gamma}\sin\psi}{\sqrt{1+\bar{\gamma}^2}} - \bar{\gamma}(\sin\psi + \bar{\gamma}\cos\psi)\right) + \sin s_2\left(\sqrt{1+\bar{\gamma}^2}\sin\psi + \bar{\gamma}\sqrt{1+\bar{\gamma}^2}\cos\psi\right)$$
$$\frac{\partial\theta}{\partial\tau} = 20\left(\cos s_2\frac{\cos\psi - \bar{\gamma}\sin\psi}{\sqrt{1+\bar{\gamma}^2}} + \sqrt{1+\bar{\gamma}^2}\sin s_2\sin\psi\right)$$
$$\frac{\partial s_2}{\partial\tau} = \frac{\cos\psi - \bar{\gamma}\sin\psi}{\sqrt{1+\bar{\gamma}^2}}\cos s_2 + \left(\sqrt{1+\bar{\gamma}^2}\sin\psi + 2\bar{\gamma}\frac{(\cos\psi - \bar{\gamma}\sin\psi)}{\sqrt{1+\bar{\gamma}^2}}\right)\sin s_2.$$

The first and the last equation define an autonomous differential equation. We conjecture that, generically on $\bar{\gamma}$, this differential equation will have at most a countable number of nondegenerate peridic orbits.

In order to have periodic orbits in (ψ,θ,s_1), we need the additional condition

$$\begin{cases} 2k\pi = 20\int_0^T \left(\cos s_2\frac{\cos\psi - \bar{\gamma}\sin\psi}{\sqrt{1+\bar{\gamma}^2}} + \sqrt{1+\bar{\gamma}^2}\sin s_2\sin\psi\right)d\tau \\ k\in\mathbb{Z} \end{cases}.$$

We conjecture that, generically on $\bar{\gamma}$, this condition is not satisfied, hence that there are no periodic orbits in (ψ,θ,s_2).

We cannot rule out other periodic orbits which would be partly made of orbits of ξ_N continued by orbits of ξ.

Such orbits have $|\Delta\theta| \geq C > 0$ since we can assume that there are no periodic orbits of ξ closing up near a repelling or an attractive orbit of v.

We claim that:

Proposition 3′. $|\Delta\theta| \geq 2\pi$ *on such periodic orbits.*

Proof. We come back to the differential equations corresponding to the flow of ξ_N. We first claim that an orbit of ξ_N, under the energy bound, cannot go from the inner boundary of the torus of modification to the outer boundary of this torus. Indeed, we have

$$\dot{s}_2 = O\left(\frac{1}{\rho\bar{\gamma}}\right).$$

Thus,

$$|\Delta s_2| = |O\left(\frac{1}{\rho\bar{\gamma}}\right)| = |-\int \frac{ds_1}{\varphi}| \geq \frac{2\pi N}{|I_0|} \times \frac{|I_0|}{2} = \pi N$$

which is impossible for N large enough.

Thus, a piece of orbit of ξ_N which contributes to a periodic orbit of the contact vector-field goes from the outer boundary to the outer boundary , i.e. from $s_2 = 0$ to $s_2 = 0$ and has

$$|\Delta\theta| \geq C > 0.$$

Coming back to the equations defining the flow of ξ_N, we find

$$\dot{s}_2 = \frac{\dot{\theta}}{20} + O\left(\frac{1}{\rho}\right).$$

Thus,

$$0 = \Delta s_2 = \frac{\Delta\theta}{20} + O\left(\frac{\Delta t}{\rho}\right)$$

and this implies that

$$|\Delta t| \geq C_\rho.$$

Hence, since $\frac{\partial}{\partial\tau} = \rho\bar{\gamma}\frac{\partial}{\partial t}$

$$\Delta\tau \geq \frac{C_\rho}{\rho\bar{\gamma}} = \frac{C}{\bar{\gamma}}.$$

In the variable τ

$$\frac{\partial\theta}{\partial\tau} = 20\cos(s_2 - \psi) + O(\bar{\gamma})$$

$$\frac{\partial}{\partial\tau}(s_2 - \psi) = \bar{\gamma}\sin(\psi + s_2) + O(\bar{\gamma}^2).$$

We start at $s_2 = 0$ and we have an interval of time at least equal to $\frac{C}{\bar{\gamma}}$ ahead of us. Either $|\cos(s_2 - \psi)|$ or $|\sin(\psi + s_2)|$ is therefore larger than $\frac{1}{2}$ as we start.

Assume first that $|\cos(s_2 - \psi)| \geq \frac{1}{2}$ as we start. Since $\frac{\partial}{\partial\tau}(s_2 - \psi) = O(\bar{\gamma}), |\cos(s_2 - \psi)|$ will remain larger than $\frac{1}{4}$ for a time interval I of length larger than $\frac{c_1}{\bar{\gamma}}$.

Thus, taking I of length $c_1/\bar\gamma$,

$$|\Delta\theta| = 2\theta\Big|\int_I \cos(s_2-\psi) + O(\bar\gamma)|I|\Big| \geq \frac{c_1}{4\bar\gamma} - C$$

and the conclusion follows.

The argument extends to the case where, at any time τ,

$$|\cos(s_2-\psi)| \geq c > 0$$

with c any prescribed positive constant ($\bar\gamma$ small in relation to c).

Thus, we may assume that

$$|\cos(s_2-\psi)| \leq c \quad \text{small}$$

on the entire piece of ξ_N-orbit.

Then, we have

$$\frac{\partial}{\partial\tau}\left(\psi - \frac{\theta}{20}\right) = \bar\gamma\sin(s_2-\psi) + O(\bar\gamma^2)$$

and

$$\left(\psi - \frac{\theta}{20}\right)(\tau) = \left(\psi - \frac{\theta}{20}\right)(0) \pm \bar\gamma\tau(1+0(1)).$$

Also

$$\begin{aligned}\frac{\partial}{\partial\tau}(s_2-\psi) &= \bar\gamma\sin(\psi+s_1) + O(\bar\gamma^2) = \bar\gamma(\sin 2\psi\cos(s_2-\psi) + \cos 2\psi\sin(s_2-\psi)) + O(\bar\gamma^2)\\ &= \pm\bar\gamma\cos 2\psi + O(\bar\gamma).\end{aligned}$$

Observe that the constraint $|\cos(s_2-\psi)| \leq c$ forces

$$|s_2 - \psi + \frac{(2k+1)\pi}{2}| \leq c.$$

Thus, at the entry and at the exit point,

$$\psi = \frac{(2k+1)\pi}{2} \pm c$$

and

$$\begin{aligned}x &= \rho\cos\left(\frac{(2k+1)\pi}{2} \pm c\right)\\ y &= \rho\sin\left(\frac{(2k+1)\pi}{2} \pm c\right).\end{aligned}$$

After integration, we derive

$$\int_I \frac{\partial}{\partial \tau}(s_2 - \psi) = \pm\bar{\gamma}\int_I \cos 2\psi + o(\bar{\gamma})|I|.$$

Since $\Delta(s_2 - \psi) = o(1)$, we must have

$$\int_I \cos 2\psi = o(|I|)$$

for any I such that

$$|I| \geq \frac{c}{\bar{\gamma}}.$$

Otherwise, there will be an I, with $|I| \geq \frac{C}{\bar{\gamma}}$, such that

$$\left| \pm\bar{\gamma}\int_I \cos 2\psi + o(\bar{\gamma})|I| \right| \geq c_1\bar{\gamma}|I| \geq c_1 C$$

yielding a contradiction.

Hence, on any such interval I, there exists an integer q and a certain time τ_1 if I such that

$$2\psi + \frac{(2q+1)\pi}{2} = o(1).$$

Thus,

$$\psi + \frac{(2q+1)\pi}{4} = o(1).$$

Comparing, we derive that

$$|\psi(0) - \psi(\tau_1)| \geq \frac{\pi}{4}(1 + o(1)).$$

Thus,

$$|\frac{\theta}{20}(\tau_1) - \frac{\theta}{20}(0) \pm \bar{\gamma}\tau_1(1 + o(1)| \geq \frac{\pi}{4}(1 + o(1)).$$

If τ is the entire time spent on this ξ_N-piece of orbit, either

$$|\bar{\gamma}\tau| \geq \frac{\pi}{8}(1 + o(1))$$

which forces

$$|\theta(\tau) - \theta(0)| \geq 20 \cdot \frac{\pi}{8}(1 + o(1)) \geq 2\pi$$

since

$$\psi(\tau) - \psi(0) = o(1).$$

Or

$$|\bar{\gamma}\tau| \leq \frac{\pi}{8}(1+o(1)).$$

Thus,

$$|\bar{\gamma}\tau_1| \leq \frac{\pi}{8}(1+o(1)).$$

Thus,

$$|\theta(\tau_1)-\theta(0)| \geq 20\frac{\pi}{8}(1+o(1)) \geq 2\pi \quad \text{again.}$$

□

Conformal deformation.

Let λ be a positive function on M. We consider the contact form $\lambda\alpha_N$ where α_N is α modified by the construction of this large rotation.

We assume that

$$d\alpha_N(v_N,[\xi,v_N]) = -1 \quad \text{with } v_N = \varphi_N v$$

in the region of M where we will carry out our constructions and computations. For simplicity and generality, we come back here to the following notations

$$v \text{ instead of } v_N$$

$$\xi_0 \text{ instead of } \xi_N$$

$$\alpha_0 \text{ instead of } \alpha_N.$$

It must though be kept clear to the mind that in the application below - which is our main purpose - $v=\varphi_N v, \xi_N=\xi_0, \alpha_N=\alpha_0$. Later, we will have v_N and v, ξ_N and ξ_0, α_N and α_0. This is why we want to avoid any confusion.

α is $\lambda\alpha_0$ (it will be $\lambda\alpha_N$ thereafter).

We thus assume that

$$d\alpha_0(v,[\xi_0,v]) = -1$$

in the region of M where we will carry out out constructions and computations.

We start with

Lemma 8. $\xi = \frac{\xi_0}{\lambda} + \frac{d\lambda(v)}{\lambda^2}[\xi_0,v] - \frac{d\lambda([\xi_0,v])}{\lambda^2}v.$

Proof. We compute

$$(d\lambda\wedge\alpha_0+\lambda d\alpha_0)(\xi,v) = (d\lambda\wedge\alpha_0)(\xi,v)+\lambda d\alpha_0(\xi,v) =$$

$$= -\frac{d\lambda(v)}{\lambda}+\frac{d\lambda(v)}{\lambda}d\alpha_0([\xi_0,v],v) = 0$$

$$(d\lambda\wedge\alpha_0+\lambda d\alpha_0)(\xi,[\xi_0,v]) = (d\lambda\wedge\alpha_0)(\xi,[\xi_0,v])+\lambda d\alpha_0(\xi,[\xi_0,v]) =$$

$$= -\frac{d\lambda([\xi_0,v])}{\lambda} - \frac{d\lambda([\xi_0,v])}{\lambda} d\alpha_0(v,[\xi_0,v]) = 0.$$

□

We now compute

$$d\alpha(v,[\xi,v]) = \lambda d\alpha_0(v,[\xi,v]).$$

Lemma 9.

$$-\gamma = d\alpha(v,[\xi,v]) = -\lambda\left(\frac{1}{\lambda} + \left(\frac{1}{\lambda}\right)_{vv} + \left(\frac{1}{\lambda}\right)_{v} d\alpha_0(v,[[\xi_0,v],v])\right).$$

Proof.

$$\lambda d\alpha_0(v,[\xi,v]) = \lambda d\alpha_0\left(v,\left[\frac{\xi_0}{\lambda} + \frac{d\lambda(v)}{\lambda^2}[\xi_0,v] - \frac{d\lambda([\xi_0,v])}{\lambda^2}v, v\right]\right) =$$

$$= d\alpha_0(v,[\xi_0,v]) - \lambda\left(\frac{d\lambda(v)}{\lambda^2}\right)_v d\alpha_0(v,[\xi_0,v]) + \frac{d\lambda(v)}{\lambda} d\alpha_0(v,[[\xi_0,v],v])$$

$$= -\lambda\left(\frac{1}{\lambda} + \left(\frac{1}{\lambda}\right)_{vv} + \left(\frac{1}{\lambda}\right)_{v} d\alpha_0(v,[[\xi_0,v],v])\right).$$

□

Corollary 3. *Set $\lambda_t = \frac{1}{t+\frac{1-t}{\lambda}}$. If $d\alpha(v,[\xi,v])(x) < 0$, then so is $d\alpha_t(v,[\xi_t,v])(x)$ for $\alpha_t = \lambda_t\alpha_0$.*

Proof. $d\alpha_t(v,[\xi_t,v])(x) = -\lambda_t\left(\frac{1-t}{\lambda} + t + \left(\frac{1-t}{\lambda}\right)_{vv} + \left(\frac{1-t}{\lambda}\right)_v d\alpha_0(v,[[\xi_0,v],v])\right)$ and the result follows. □

Assume now that

$$\lambda\left(\frac{1}{\lambda}\right)_v, \lambda\left(\frac{1}{\lambda}\right)_{vv} \text{ are } o(1).$$

Recall that

$$\gamma(x) = 1 + \lambda\left(\frac{1}{\lambda}\right)_{vv} + \lambda\left(\frac{1}{\lambda}\right)_v d\alpha_0(v,[[\xi_0,v],v])$$

and

$$\tilde{v} = \frac{v}{\sqrt{\gamma(x)}}$$

so that

$$d\alpha(\tilde{v},[\xi,\tilde{v}]) = \frac{1}{\gamma(x)} d\alpha(v,[\xi,v]) = -1.$$

We compute in the sequel

$$\tilde{\bar{\mu}} = d\alpha(\tilde{v},[\tilde{v},[\xi,\tilde{v}]])$$
$$\tilde{\bar{\mu}}_\xi = d\tilde{\bar{\mu}}(\xi)$$
$$\tilde{\bar{\mu}}_v = d\tilde{\bar{\mu}}(v)$$

and also $\tilde{\tau}$, where $[\xi,[\xi,\tilde{v}]] = -\tilde{\tau}\tilde{v}$.

Lemma 10. $\tilde{\tilde{\mu}} = \frac{1}{\gamma(x)^{3/2}} \left(d\alpha_0(v,[v,[\xi_0,v]]) \left(1 + 2\lambda \left(\frac{1}{\lambda}\right)_{vv} + \frac{d\lambda(v)}{\lambda}(2+\gamma(x)) - \lambda \left(\frac{1}{\lambda}\right)_{vvv} + \right.\right.$ $\left. + \frac{\lambda_v}{\lambda} d\alpha_0(v,[v,[[\xi_0,v],v]]\right)$.

Proof. Clearly,

$$\tilde{\tilde{\mu}} = \frac{1}{\gamma(x)^{3/2}} d\alpha(v,[v,[\xi,v]]).$$

We have

$$d\alpha(v,[v,[\xi,v]]) = (d\lambda \wedge \alpha_0 + \lambda d\alpha_0)(v,[v,[\xi,v]]) = d\lambda(v)\alpha_0([v,[\xi,v]]) +$$

$$+\lambda d\alpha_0(v,[v,[\xi,v]]) = \gamma(x)\frac{d\lambda(v)}{\lambda} + \lambda d\alpha_0\left(v,\left[v,\left[\frac{\xi_0}{\lambda},v\right]\right]\right) + \lambda d\alpha_0\left(v,\left[v,\left[\frac{d\lambda(v)}{\lambda^2}[\xi_0,v],v\right]\right]\right) =$$

$$= \gamma(x)\frac{d\lambda(v)}{\lambda} - \lambda\left(\frac{1}{\lambda}\right)_v d\alpha_0(v,[v,\xi_0]) + \lambda d\alpha_0\left(v,\left[v,\frac{1}{\lambda}[\xi_0,v]\right]\right) +$$

$$+\lambda d\alpha_0\left(v,\left[v,\left[\frac{d\lambda(v)}{\lambda^2}[\xi_0,v],v\right]\right]\right) = d\alpha_0(v,[v,[\xi_0,v]]) - 2\lambda\left(\frac{1}{\lambda}\right)_v d\alpha_0(v,[v,\xi_0]) +$$

$$+\gamma(x)\frac{d\lambda(v)}{\lambda} + \lambda d\alpha_0\left(v,\left[v,\left(\frac{1}{\lambda}\right)_{vv}[\xi_0,v]\right]\right) + \frac{\lambda_v}{\lambda}d\alpha_0(v,[v,[[\xi_0,v],v]])$$

$$-\lambda\left(\frac{1}{\lambda}\right)_{vv} d\alpha_0(v,[[\xi_0,v],v]) = d\alpha_0(v,[v,[\xi_0,v]])\left(1+2\lambda\left(\frac{1}{\lambda}\right)_{vv}\right) - \lambda\left(\frac{1}{\lambda}\right)_{vvv}$$

$$-2\lambda\left(\frac{1}{\lambda}\right)_v + \gamma(x)\frac{d\lambda(v)}{\lambda} + \frac{\lambda_v}{\lambda}d\alpha_0(v,[v,[[\xi_0,v],v]]).$$

□

Observe that

$$\lambda\left(\frac{d\lambda(v)}{\lambda^2}\right)_v = \frac{\lambda_{vv}}{\lambda} - 2\frac{\lambda_v^2}{\lambda^2}$$
$$\lambda\left(\frac{d\lambda(v)}{\lambda^2}\right)_{vv} = \frac{\lambda_{vvv}}{\lambda} - 5\frac{\lambda_v\lambda_{vv}}{\lambda^2} + 4\frac{\lambda_v^3}{\lambda^3}.$$

Next, we compute $\tilde{\tau}$. We know that $-\tilde{\tau}$ is the collinearity coefficient of $[\xi,[\xi,\tilde{v}]]$ on $\tilde{v}$.

We will therefore compute $[\xi,[\xi,\tilde{v}]]$ in the $(\xi_0, v, [\xi_0,v])$ basis and we will track down the component on v, throwing away the other components.

We have

Lemma 11. *Let $\nu(x) = \sqrt{\gamma(x)}$. Then,*

$$\tilde{\tau} = -\frac{1}{\nu(x)}\left[\frac{dA(\xi_0)}{\lambda} - \frac{B\tau}{\lambda} + \frac{d\lambda(v)}{\lambda^2}dA([\xi_0,v]) - A\bar{\mu}_{\xi_0}\frac{d\lambda(v)}{\lambda^2} - \frac{d\lambda([\xi_0,v])}{\lambda^2}dA(v) + \right.$$

$$\left. + A\left(\frac{d\lambda([\xi_0,v])}{\lambda^2}\right)_v - B\frac{d\lambda([\xi_0,v])}{\lambda^2}\bar{\mu}_{\xi_0} + B\left(\frac{d\lambda([\xi_0,v])}{\lambda^2}\right)_{[\xi_0,v]}\right] \text{ with}$$

$$A = d\nu(\xi) - \bar{\mu}_{\xi_0}\nu\frac{d\lambda(v)}{\lambda^2} + \nu\left(\frac{d\lambda([\xi_0,v])}{\lambda^2}\right)_v, B = \nu\left(\frac{1}{\lambda} + \bar{\mu}\frac{d\lambda(v)}{\lambda^2} + \left(\frac{1}{\lambda^2}\right)_{vv}\right).$$

Proof.

$$[\xi,[\xi,\tilde{v}]] = \left[\frac{\xi_0}{\lambda} + \frac{\lambda_v}{\lambda^2}[\xi_0,v] - \frac{d\lambda([\xi_0,v])}{\lambda^2}v, \left[\frac{\xi_0}{\lambda} + \frac{\lambda_v}{\lambda^2}[\xi_0,v] - \frac{d\lambda([\xi_0,v])}{\lambda^2}v, \tilde{v}\right]\right]$$

with $\tilde{v} = \nu(x)v$. Observe that

$$[v,[\xi_0,v]] = \xi_0 - \bar{\mu}[\xi_0,v] + \bar{\mu}_{\xi_0}v.$$

Indeed,

$$\begin{aligned} &\alpha_0([v,[\xi_0,v]]) = 1 \\ &d\alpha_0([v,[\xi_0,v]],v) = -\bar{\mu} \\ &d\alpha_0([\xi_0,v],[v,[\xi_0,v]]) = \bar{\mu}_{\xi_0} = d\bar{\mu}(\xi_0). \end{aligned}$$

We then compute

$$[\xi,\tilde{v}] = d\nu(\xi)v + \nu\left(-\left(\left(\frac{1}{\lambda}\right)_v + \frac{\lambda_v}{\lambda^2}\right)\xi_0 + [\xi_0,v]\left(\frac{1}{\lambda} + \left(\frac{1}{\lambda}\right)_{vv} + \bar{\mu}\frac{\lambda_v}{\lambda^2}\right) + \right.$$

$$\left. + v\left(\bar{\mu}_{\xi_0}\left(\frac{1}{\lambda}\right)_v + \left(\frac{d\lambda([\xi_0,v])}{\lambda^2}\right)_v\right)\right) = v\left(d\nu(\xi) + \bar{\mu}_{\xi_0}\left(\frac{1}{\lambda}\right)_v + \nu\left(\frac{d\lambda([\xi_0,v])}{\lambda^2}\right)_v\right) +$$

$$+[\xi_0,v]\nu\left(\frac{1}{\lambda} + \left(\frac{1}{\lambda}\right)_{vv} + \bar{\mu}\frac{d\lambda(v)}{\lambda^2}\right) = Av + B[\xi_0,v].$$

Set $[\xi_0,[\xi_0,v]] = -\tau v$

$$[\xi,[\xi,\tilde{v}]] = \left[\frac{\xi_0}{\lambda} + \frac{\lambda_v}{\lambda^2}[\xi_0,v] - \frac{d\lambda([\xi_0,v])}{\lambda^2}v, Av + B[\xi_0,v]\right] = v\left(\frac{dA(\xi_0)}{\lambda} - \frac{B\tau}{\lambda} + \right.$$

$$\frac{\lambda_v}{\lambda^2}dA([\xi_0,v]) - \frac{d\lambda([\xi_0,v])}{\lambda^2}dA(v) - \bar{\mu}_{\xi_0}A\frac{\lambda_v}{\lambda^2} + A\left(\frac{d\lambda([\xi_0,v])}{\lambda^2}\right)_v + B\left(\frac{d\lambda([\xi_0,v])}{\lambda^2}\right)_{[\xi_0,v]} -$$

$$-B\bar{\mu}_{\xi_0}\frac{d\lambda([\xi_0,v])}{\lambda^2}\Bigg).$$

Thus, $\tilde{\tau} = -\frac{1}{\nu}\left(\frac{dA(\xi_0)}{\lambda} - \frac{B\tau}{\lambda} + \frac{d\lambda(v)}{\lambda^2}dA([\xi_0,v]) - A\bar{\mu}_{\xi_0}\frac{d\lambda(v)}{\lambda^2} - \frac{d\lambda([\xi_0,v])}{\lambda^2}dA(v) + A\left(\frac{d\lambda([\xi_0,v])}{\lambda^2}\right)_v + \right.$
$\left. +B\left(\frac{d\lambda([\xi_0,v])}{\lambda^2}\right)_{[\xi_0,v]} - B\bar{\mu}_{\xi_0}\frac{d\lambda([\xi_0,v])}{\lambda^2}\right).$ □

Coming back to α_N, ξ_N, v_N, we observe that $\bar{\mu}_N$ is bounded as well as its derivative since $\bar{\mu}_N$ is identically zero wherever we introduce the large, $2\pi N$ rotation and $\bar{\mu}_N$ equals $\bar{\mu}_0$ outside of the set where this modification occurs. Similarly, τ_N is bounded as well as its derivatives independently of N by construction.

Proposition 4. *$|\bar{\mu}_N| + |d\bar{\mu}_N| + |\tau_N| \leq C$, where C does not depend on N.*

Next, we show how to build λ so that $\tilde{\tau}$ remains bounded and "mountains" are built around the hyperbolic orbit. These mountains keep the variations away from this hyperbolic orbit. Since this is a quite surprising result, we complete our construction carefully.

Choice of λ.

The construction of $\xi_{N,\lambda}, \alpha_{N,\lambda} = \lambda\alpha_N$ involves the definition of the function λ. We would like to choose this function carefully with respect to $v(v_N)$ so that $\lambda d\alpha_N(v_N, [\xi_{N,\lambda}, v_N]), \tilde{\bar{\mu}}_N$, $\tilde{\bar{\mu}}_{N,v} = d\tilde{\bar{\mu}}_N(\tilde{v}_N), \tilde{\bar{\mu}}_{N,\xi_{N,\lambda}} = d\tilde{\bar{\mu}}_N(\xi_{N,\lambda}), \tilde{\tau}_N$ enjoy appropriate bounds.

To avoid unnecessary complicated notations, we again use $\xi, \alpha, \xi_0, \alpha_0$ etc. The main issue is that we cannot hope that $\beta = d\alpha(v,\cdot)$ is a contact form (with the same orientation than α).

Assuming that v is nonsingular, it is reasonable to first consider the case when the ω-limit set of v is made of periodic orbits only. Around "elliptic" (attractive or repelling) periodic orbits, $\ker\alpha$ "turns well" (see [1] p 26) so that the existence of such a β (with appropriate choices of λ) follows.

Around hyperbolic periodic orbits, $\ker\alpha$ "turns well" in most of the cases, except for one case which yields a precise (local) normal form of α and v (see Appendix 1). Then, locally, $\ker\alpha_0$ behaves (nearly) as a foliation. There is no hope for such a λ and such a β to exist near such orbits, with this behavior of α_0 and v.

We thus need to keep our homology away from such neighborhoods or to extend it using the ideas of [3], Chapter V.1 These ideas can be pushed and worked out. They still require a certain amount of work to become practical. We explore here another direction: we aim at keeping the unstable manifolds of the periodic orbits of ξ_0 away from such neighborhoods by creating "mountains" around them. These "mountains" are built by increasing to a high value the Hamiltonian λ around them so that the curves on the unstable manifolds of the periodic orbits of ξ_0 are unable to penetrate them.

We need for this a lot of rotation of $ker\alpha$ around v. This will allow us to keep control of $\tilde{\bar{\mu}}, \tilde{\tau}, d\alpha(v,[\xi,v])$ and derivatives with respect to $v, [\xi_0, v]$... We cannot get such a rotation from the neighborhood of a "bad" hyperbolic orbit since $\ker\alpha_0$ turns very little around v in such a neighborhood. We have to seek for it in the neighborhood of attractive or repelling orbits and "bring it back" to our neighborhoods.

For sake of simplicity, we will assume in a first step that the stable and unstable manifolds of our "bad" hyperbolic orbit θ do not intersect the unstable and stable manifolds of another hyberbolic orbit. We will discuss the case later.

Thus,

$(\mathcal{H}1)$ The stable (respectively unstable) manifold of $\mathcal{O}$ is part of the unstable manifold of the repelling (respectively attracting) periodic orbit.

We will assume - a very natural hypothesis which we will see to hold after a minor modification of $\ker \alpha_0$ and v if needed - that

$(\mathcal{H}2)$ α_0, v have the normal forms provided in (α_0), (v) above near the repelling and attracting orbit.

The construction of the function λ is ultimately quite involved. However, in order to describe a basic step in this direction, we first take the following example. The construction is refined later.

1st step in the construction of λ.

Let $\mathcal{W}_u(\mathcal{O})$ be unstable manifold of $\mathcal{O}$ and let $\partial\mathcal{V}$ be the boundary of a small basin for the attractive orbit. $\partial\mathcal{V}$ is a section to v. We consider $\mathcal{W}_u(\mathcal{O}) \cap \partial\mathcal{V} = \Gamma$ and a small neighborhood $\mathcal{L}$ of it in $\partial\mathcal{V}$. Let $\mathcal{L}'$ be an even smaller neighborhood, $\theta = \theta(x_0), x_0 \in \mathcal{L}$ be a C^∞- function valued in [0,1], equal to 1 on $\mathcal{L}'$ and to zero outside of $\mathcal{L}$.

We set:

$$\lambda(x) = e^{\delta\theta(x_0)s(\tau)}.$$

for all x of $W_s(\mathcal{L})$ i.e. for all the $x's$ of the flow-lines of v abuting in $\mathcal{L}$.

Such $x's$ are parametrized by a base point x_0 in $\mathcal{L}$ and a time τ on the (reverse) flow-line of v abuting at x_0. δ is a small number which we will choose later.

More generally,

$$\lambda(x) = e^{\delta(\sum \theta_i(x_i)s_i(\tau_i))}$$

where the (x_i, τ_i) are various sets of parameters traking a point x of M through its reference point x_i in a section to v and a time τ_i on the flow-line of v abuting at x_i.

Clearly

$$\frac{\lambda_v}{\lambda} = \delta\left(\sum \theta_i \frac{\partial s_i}{\partial \tau_i}\right), \frac{\lambda_{vv}}{\lambda} = \delta \sum \theta_i \frac{\partial^2 s_i}{\partial \tau_i^2} + \delta^2\left(\sum \theta_i \frac{\partial s_i}{\partial \tau_i}\right)^2$$

$$\lambda\left(\frac{1}{\lambda}\right)_{vvv} = \left(\lambda\left(\frac{1}{\lambda}\right)_{vv}\right)_v - \lambda_v\left(\frac{1}{\lambda}\right)_{vv} = \left(\lambda\left(\frac{1}{\lambda}\right)_{vv}\right)_v - \frac{\lambda_v}{\lambda}\lambda\left(\frac{1}{\lambda}\right)_{vv} =$$
$$= \left(\left(\lambda\left(\frac{1}{\lambda}\right)_v\right)_v - \frac{\lambda_v}{\lambda}\lambda\left(\frac{1}{\lambda}\right)_v\right)_v - \frac{\lambda_v}{\lambda}\lambda\left(\frac{1}{\lambda}\right)_{vv} =$$
$$= \delta O\left(\sum \theta_i\left(\left|\frac{\partial^3 s_i}{\partial \tau_i^3}\right| + \left|\frac{\partial s_i}{\partial \tau_i}\right|\left|\frac{\partial^2 s_i}{\partial \tau_i^2}\right|\left|\frac{\partial^2 s_i}{\partial \tau_i^2}\right| + \left|\frac{\partial s_i}{\partial \tau_i}\right|^3\right)\right). \tag{29}$$

We assume that

$$\left|\frac{\partial s_i}{\partial \tau_i}\right| + \left|\frac{\partial^2 s_i}{\partial \tau_i^2}\right| + \left|\frac{\partial^3 s_i}{\partial \tau_i^3}\right| = O(1). \tag{30}$$

Then,

$$\frac{\lambda_v}{\lambda}, \lambda\left(\frac{1}{\lambda}\right)_{vv}, \lambda\left(\frac{1}{\lambda}\right)_{vvv} \text{ are } O(\delta).$$

In this way, $\gamma(x)$ and $\tilde{\bar{\mu}}$ are under control. We need to worry about $d\tilde{\bar{\mu}}$ and $\tilde{\tau}$. Coming back to the formula of $\tilde{\tau}$ in Lemma 11, to A and B as well a the formulae for ν and ξ, we see that these formulae involve derivatives of $\frac{1}{\lambda}$. According to our choice of λ above, λ is larger than or equal to 1, might tend to $+\infty$. Because we are only considering negative powers of λ and derivatives of such quantities, we do not fear the increase of λ to infinity.

The derivative of $\delta \sum \theta_i(x_i)s_i(\tau_i)$ yield more problems because s_i may be very large and derivatives of θ may also be very large. Since we want λ to be very large when $\theta_i = 1$ and we are at the "end" of the (reverse) flow-lines, we require

$$\delta s_i(\bar{\tau}_i) = \text{ Log } \bar{\lambda}$$

where $\bar{\lambda}$ is some large number. The flow-lines are defined on $[o, \bar{\tau}_i]$. Thus δ cannot tame $s_i(\tau_i)$.

We observe that, as we modify α_0 into α_N and ξ_0 in ξ_N, see Proposition 2,

$$\bar{\mu}_N = d\alpha_N(v_N, [v_N, [\xi_N, v_N]])$$

is zero in the domain where the modification takes place.

Also, since in this domain $[v_N, [v_N, \xi_N]] = -\xi_N$,

$$d\alpha_N(v_N, [v_N, [[\xi_N, v_N], v_N]]) = -d\alpha_N(v_N, [v_N, \xi_N]) = -1.$$

Thus, in the domain where ξ_0 is modified into ξ_N

$$\begin{cases} \bar{\mu}_N = 0 \\ d\alpha_N(v_N, [v_N, [[\xi_N, v_N], v_N]] = -1 \\ \gamma_N(x) = 1 + \lambda\left(\frac{1}{\lambda}\right)_{v_N v_N} \\ \tilde{\bar{\mu}}_N = \frac{1}{\gamma_N^{3/2}}\left(\frac{d\lambda(v_N)}{\lambda}(2 + \gamma_N(x)) - \lambda\left(\frac{1}{\lambda}\right)_{v_N v_N v_N} - \frac{\lambda_{v_N}}{\lambda}\right). \end{cases} \tag{31}$$

We then have

Proposition 5. *Assume s_i is only a function of τ_i, the time along v_N.*

Then, there exists C independent of $N, \delta, \bar{\lambda}$ such that, given N and $\bar{\lambda}$,

$$|\tilde{\bar{\mu}}_N| + |d\tilde{\bar{\mu}}_N(\xi_N)| + |d\tilde{\bar{\mu}}_N(v_N)| + |d\tilde{\bar{\mu}}_N([\xi_N, v_N])| + |d\tilde{\bar{\mu}}_N(\xi_{N,\lambda})| +$$

$$d\tilde{\bar{\mu}}_N([\xi_{N,\lambda}, v_N]) + |d(d\tilde{\bar{\mu}}_N(\xi_{N,\lambda}))(\xi_{N,\lambda})| \leq C$$

Proof. Recall that $\lambda = e^{\delta(\sum \theta_i s_i)}$.

Either x is in the domain where ξ_0 has been modified into ξ_N.

Then, $\xi_N \cdot \xi_N, \xi_N \cdot [\xi_N, v_N], [\xi_N, v_N] \cdot \xi_N, [\xi_N, v_N] \cdot [\xi_N, v_N]$ split over e_1, e_2 with bounded coefficients, the bounds being C^1 and independent of N.

We do not claim any control on the v_N-components of these vectors. $\tilde{\bar{\mu}}_N$ is expressed using $\lambda\left(\frac{1}{\lambda}\right)_{v_N}, \lambda\left(\frac{1}{\lambda}\right)_{v_N v_N}$ and $\lambda\left(\frac{1}{\lambda}\right)_{v_N v_N v_N}$. Since the θ_i's have a zero derivative along v_N, all these expressions read as products

$$\delta\left(\sum \theta_i \frac{\partial^m s_i}{\partial \tau_i^m}\right) \quad m = 1, 2, 3.$$

By construction $ds_i(e_1) = ds_i(e_2) = 0$ since $d\tau_i(e_1) = d\tau_i(e_2) = 0$ while $d\tau_i(\varphi_N v) = 1$. $\xi_N, v_N, [\xi_N, v_N]$ split on e_1, e_2 and $\varphi_N v$ with bounded coefficients. Thus $d\theta_i(\xi_N), d\theta_i(v_N), d\theta_i$ $([\xi_N, v_N])$ are clearly bounded and $|\tilde{\bar{\mu}}_N| + |d\tilde{\bar{\mu}}(\xi_N)| + |d\tilde{\bar{\mu}}_N(v_N)| + |d\tilde{\bar{\mu}}_N\left([\xi_N, v_N]\right)|$ is bounded and even $0(\delta)$ in such a region.

For $d(d\tilde{\bar{\mu}}_N(\xi_{N,\lambda}))(\xi_{N,\lambda})$ we come back to the expression of $\xi_{N,\lambda}$

$$\xi_{N,\lambda} = \frac{\xi_N}{\lambda} + \frac{d\lambda(v_N)}{\lambda^2}[\xi_N, v_N] - \frac{d\lambda([\xi_N, v_N])}{\lambda^2} v_N.$$

We thus have to take derivatives which are typically expressions such as

$$\frac{\delta}{\lambda}\left(\sum \theta_i \frac{\partial^m s_i}{\partial \tau_i^m}\right)_{\xi_N} + \frac{\delta d\lambda(v_N)}{\lambda^2}\left(\sum \theta_i \frac{\partial^m s_i}{\partial \tau_i^m}\right)_{[\xi_N, v_N]} - \delta \frac{d\lambda([\xi_N, v_N])}{\lambda^2}\left(\sum \theta_i \frac{\partial^m s_i}{\partial \tau_i^m}\right)_{v_N}$$

and then take again a derivative of such expressions along $\xi_{N,\lambda}$. On $\frac{\partial^m s_i}{\partial \tau_i^m}$, the e_1 and e_2 components of each derivative do not give any contribution. It is only the $\varphi_N v$-components which give a contribution. These are bounded and have a bounded derivative along $\xi_N, [\xi_N, v_N], v_N$. The problems come only after taking a first derivative of θ_i and then going on with a second derivative of this expression. Typically, we need to estimate

$$(d\theta_i(\xi_N))_{\xi_N}, (d\theta_i([\xi_N, v_N]))_{\xi_N}, (d\theta_i([\xi_N, v_N]))_{v_N}, (d\theta_i([\xi_N, v_N]))_{[\xi_N, v_N]}, (d\theta_i(\xi_N))_{[\xi_N, v_N]} \text{ etc.}$$

We recall now that

$$\xi_N \cdot \xi_N, [\xi_N, v_N] \cdot [\xi_N, v_N], \xi_N \cdot [\xi_N, v_N], [\xi_N, v_N] \cdot \xi_N$$

are bounded transversally to v_N. Furthermore, $d\theta_i(v_N) = 0$. Thus,

$$(d\theta_i(\xi_N))_{\xi_N}, (d\theta_i([\xi_N, v_N]))_{\xi_N} \text{ etc.}$$

are bounded independently of N.

For $d\theta_i([\xi_N, v_N])_{v_N}$ and the like, we observe that

$$d\theta_i([\xi_N, v_N])_{v_N} = d\theta_i([v_N, [\xi_N, v_N]])$$

since $d\theta_i(v_N) = 0$ and the conclusion follows again.

Thus,

$$|d(d\tilde{\tilde{\mu}}_N(\xi_{N,\lambda})(\xi_{N,\lambda})| = O(\delta^{m+1} \sum \bar{\tau}_i^m) = \delta O(\text{ Log}^m \bar{\lambda}).$$

We can bound in a similar way $d\tilde{\tilde{\mu}}_N(\xi_{N,\lambda})$.

For $d\tilde{\tilde{\mu}}_N([\xi_{N,\lambda}, v_N])$, we observe that

$$[\xi_{N,\lambda}, v_N] = \frac{[\xi_N, v_N]}{\lambda} + \frac{d\lambda(v_N)}{\lambda^2}\xi_N + \frac{\lambda_{v_N}}{\lambda}[[\xi_N, v_N], v_N] -$$

$$- \left(\frac{\lambda_{v_N}}{\lambda}\right)_{v_N} [\xi_N, v_N] + \left(\frac{d\lambda([\xi_N, v_N])}{\lambda^2}\right)_{v_N} v_N.$$

The condition follows again since we have an additional δ coming from the expression of $\tilde{\tilde{\mu}}_N$.

Finally, if x is not in the domain where ξ_0 has been modified into ξ_N, then $d\alpha_0(v, [v, [\xi_0, v]])$, $d\alpha_0(v, [v, [\xi_0, v], v]])$ are C^∞-functions and even though $\tilde{\tilde{\mu}}$ is not expressed only with the use of $\lambda\left(\frac{1}{\lambda}\right)_v$ and other terms of the same type, $\tilde{\tilde{\mu}}$ is a product of these terms with these C^∞-functions, which are independent of N. The above argument extends verbatim.

Before proceeding with the estimate on $\tilde{\tau}$, we make the following four observations:

Observation 1 As we take a derivative of s_i along $\xi_N, [\xi_N, v_N]$ or v_N (which we see as split on the basis (e_1, e_2, v_N)), $\varphi_N v = v_N$ is absorbed in $\frac{\partial s_i}{\partial \tau_i}$ or other derivatives of the same type, but higher order $(d\theta_i(v) = 0)$. v_N gone, we are left with the coefficient of v_N which is C^1-bounded independently of N. We can take safely another derivative along $\xi_N, [\xi_N, v_N]$ or v_N. We will not hit φ_N with a derivative.

Observation 2 Since $d\theta_i(v) = 0, v_N \cdot d\theta_i(w) = d\theta_i([v_N, w])$. We may then take one more derivative along a direction such as $\xi_N, [\xi_N, v_N]$. We know that $\xi_N \cdot \xi_N, \xi_N \cdot [\xi_N, v_N], [\xi_N, v_N] \cdot \xi_N, [\xi_N, v_N] \cdot [\xi_N, v_N]$ are bounded transversally to v and that $d\theta_i(v) = 0$. We get then bounds on such expressions which depend on $|\theta_i|_{C^2}$ and are independent of N.

Observation 3 If we take a derivative of s_i along v_N, ξ_N or $[\xi_N, v_N]$, we free a δ. Furthermore, in all our computations, we never take a v_N-deriative after taking two derivatives along ξ_N or $[\xi_N, v_N]$. Otherwise, we might end up with terms such as $d\theta_i(d\varphi_N(v_N) \cdot (\xi_N \cdot v))$. This never happens.

Observation 4 Thus, if a derivative is taken along v_N, either it goes onto s_i and frees a δ, or it goes onto $d\theta_i(\xi_N)$ or $d\theta_i([\xi_N, v_N])$. Since $d\theta_i(v_N) = 0$, we end up with $d\theta_i$ of a Lie bracket ($[v_N, \xi_N]$ or $[v_N, [\xi_N, v_N]] = \xi_N$), hence with an expression of the same type.

Taking more derivatives along v_N will not change this pattern. We can then always take one more derivative along ξ_N or $[\xi_N, v_N]$ and use Observation 1 if the expression which we have contains a $d\theta_i(\xi_N)$ or $d\theta_i([\xi_N, v_N])$; or this expression contains only θ_i and we can then take two derivatives along ξ_N and/or $[\xi_N, v_N]$. The result is bounded independently of N.

Using the four observations above, we turn to $\tilde{\tau}$ and estimate it, firstly in the domain where ξ_0 has been relpaced by ξ_N.

Then, in Lemma 11, A and B reduce to

$$A = d\nu(\xi) + \nu\left(\frac{d\lambda([\xi_0, v])}{\lambda^2}\right)_v$$
$$B = \nu\left(\frac{1}{\lambda} + \left(\frac{1}{\lambda}\right)_{vv}\right)$$

and $\tilde{\tau}$ reads

$$\tilde{\tau} = -\frac{1}{\nu(x)}\left[\frac{dA(\xi_0)}{\lambda} - \frac{B\tau}{\lambda} + \frac{d\lambda(v)}{\lambda}dA([\xi_0, v]) - \frac{d\lambda([\xi_0, v])}{\lambda^2}dA(v) + \right.$$
$$\left. + A\left(\frac{d\lambda([\xi_0, v])}{\lambda^2}\right)_v + B\left(\frac{d\lambda([\xi_0, v])}{\lambda^2}\right)_{[\xi_0, v]}\right]$$

ξ_0 is in fact ξ_N and $v = v_N$. Recall that $\bar{\mu}_N$ is zero. Thus, since $\bar{\mu}_{N,\xi_N\xi_N} + \tau_N\bar{\mu}_N = -d\tau_N(v_N)[2]$, $d\tau_N(v) = 0$ and τ_N is constant on a flow-line of v. They all abut to a point where $\varphi_N = 1$ and τ_N is the original τ of α_0. Thus, τ in the expression of $\tilde{\tau}$ above is τ_N and is bounded.

$d\nu(\xi)$ is equal to

$$\frac{1}{\lambda}d\nu(\xi_0) + \frac{d\lambda(v)}{\lambda^2}d\nu([\xi_0, v]) - \frac{d\lambda([\xi_0, v])}{\lambda^2}d\nu(v).$$

All of this involves derivatives of $\lambda\left(\frac{1}{\lambda}\right)_{vv}$ along $\xi_0, [\xi_0, v], v$. In computing $\tilde{\tau}$, we take one further derivative along ξ_0 of $d\nu(\xi)$.

Taking into account our four observations, we derive that

$$(d\nu(\xi))_{\xi_0} = O(\delta \operatorname{Log}^m \bar{\lambda})$$

since the initial derivative $\lambda\left(\frac{1}{\lambda}\right)_{vv}$ frees a δ. This O depends on $d\theta_i, d^2\theta_i$.

The same estimate holds for the contribution of the $\nu\left(\frac{1}{\lambda}\right)_{vv}$ part of B. Also, taking ν-derivatives in $\nu\left(\frac{d\lambda([\xi_0,v])}{\lambda^2}\right)_v$ (the second part of A) or in $\frac{\nu}{\lambda}$ (this first part of B) yields the same estimate since $\nu = 1 + 0(\delta)$. The same holds true of $\frac{d\lambda(v)}{\lambda}dA([\xi_0, v])$.

Thus,

$$\tilde{\tau} = O(\delta \text{ Log}^m \bar{\lambda} + 1) - \left(\frac{1}{\lambda} \left(\left(\frac{d\lambda([\xi_N, v_N])}{\lambda^2} \right)_{v_N} \right)_{\xi_N} + \frac{d\lambda([\xi_N, v_N])}{\lambda^2} \left(\frac{d\lambda([\xi_N, v_N])}{\lambda^2} \right)_{v_N v_N} \right) -$$
$$- \left(\frac{d\lambda([\xi_N, v_N])}{\lambda^2} \right)^2_{v_N} - \frac{1}{\lambda} \left(\frac{d\lambda([\xi_N, v_N])}{\lambda^2} \right)_{[\xi_N, v_N]} \Bigg).$$

As we compute $\left(\frac{d\lambda([\xi_N, v_N])}{\lambda^2}\right)_{v_N}$ or $\left(\frac{d\lambda([\xi_N, v_N])}{\lambda^2}\right)_{v_N v_N}$, all v_N- derivatives have to be taken on $d\theta_i([\xi_N, v_N])$. Otherwise, a δ is freed either because v_N has been absorbed in s_i, yielding $\frac{\partial s_i}{\partial \tau_i} = O(1)$, or because $[\xi_N, v_N]$ has been applied to s_i in the first place, with the same conclusion. Such contributions can be included into $O(\delta \text{ Log}^m \bar{\lambda})$. Observe also that v_N cannot be applied to a simple θ_i since $d\theta_i(v_N) = 0$.

Thus, since $d\theta_i(v_N) = 0$

$$\left(\frac{d\lambda([\xi_N, v_N])}{\lambda^2} \right)_{v_N} = \frac{(d\lambda([\xi_N, v_N]))_{v_N}}{\lambda^2} + O(\delta \text{ Log}^m \bar{\lambda})$$

and since $v_N \cdot d\theta([\xi_N, v_N]) = d\theta_i([v_N, [\xi_N, v_N]]) = d\theta_i(\xi_N)$,

$$\left(\frac{d\lambda([\xi_N, v_N])}{\lambda^2} \right)_{v_N} = \frac{d\lambda(\xi_N)}{\lambda^2} + O(\delta \text{ Log}^m \bar{\lambda}).$$

(In $d\lambda(\xi_N)$, either the ξ_N-derivative is taken on θ_i or, if not, a δ is freed. The additional contribution is thrown into $O(\delta \text{ Log}^m \bar{\lambda})$).

Similarly,

$$\left(\frac{d\lambda([\xi_N, v_N])}{\lambda^2} \right)_{v_N v_N} = \frac{d\lambda([v_N, \xi_N])}{\lambda^2} + O(\delta \text{ Log}^m \bar{\lambda}).$$

Thus,

$$\tilde{\tau} = - \left(\frac{1}{\lambda} \left(\frac{d\lambda(\xi_N)}{\lambda^2} \right)_{\xi_N} + \frac{1}{\lambda} \left(\frac{d\lambda([\xi_N, v_N])}{\lambda^2} \right)_{[\xi_N, v_N]} - \left(\frac{d\lambda(\xi_N)}{\lambda^2} \right)^2 - \left(\frac{d\lambda([\xi_N, v_N])}{\lambda^2} \right)^2 \right) +$$
$$+ O(\delta \text{ Log}^m \bar{\lambda} + 1).$$

Observe now that

$$-\frac{1}{\lambda} \left(\frac{d\lambda(\xi_N)}{\lambda^2} \right)_{\xi_N} = -\frac{1}{\lambda^2} \left(\frac{d\lambda(\xi_N)}{\lambda} \right)_{\xi_N} + \left(\frac{d\lambda(\xi_N)}{\lambda^2} \right)^2$$

$$-\frac{1}{\lambda}\left(\frac{d\lambda([\xi_N,v_N])}{\lambda^2}\right)_{[\xi_N,v_N]}=-\frac{1}{\lambda^2}\left(\frac{d\lambda([\xi_N,v_N])}{\lambda}\right)_{[\xi_N,v_N]}+\left(\frac{d\lambda([\xi_N,v_N]}{\lambda^2}\right)^2.$$

Thus, using the identities above and the form observations stated earlier

$$\tilde{\tau}=O(\delta\ \mathrm{Log}^m\bar{\lambda}+1)-\frac{1}{\lambda^2}\left(\left(\frac{d\lambda(\xi_N)}{\lambda}\right)_{\xi_N}+\left(\frac{d\lambda([\xi_N,v_N])}{\lambda}\right)_{[\xi_N,v_N]}\right)=$$

$$=O(\delta\ \mathrm{Log}^m\bar{\lambda}+1)-e^{-2\delta\sum\theta_is_i}\sum_i\delta s_i\left(\sum\left(d\theta_i([\xi_N,v_N])\right)_{[\xi_N,v_N]}+\left(d\theta_i(\xi_N)\right)_{\xi_N}\right).$$

Let π be the pull back map from x towards the torus T_2-T_1 (outside T_1, inside of T_2) where the modification takes place (the introduction of the large rotation) onto $\partial T_2\cdot\theta_i$ is in fact defined on ∂T_2 and should be thought of as $\theta_i\circ\pi$. Thus,

$$\tilde{\tau}=O(\delta\ \mathrm{Log}^m\bar{\lambda}+1)-e^{-2\delta\sum\theta_is_i}\sum_i\delta s_i\left([\xi_N,v_N]\cdot(d\theta_i\circ d\pi)([\xi_N,v_N])+\xi_N\cdot(d\theta_i\circ d\pi)(\xi_N)\right).$$

We then split

$$[\xi_N,v_N]=d\pi([\xi_N,v_N])+\alpha_Nv_N$$

$$\xi_N=d\pi(\xi_N)+\beta_Nv_N$$

α_N and β_N are easily seen to be bounded and independent of N.

Thus, since $d\theta_i\circ d\pi(v_N)=0$,

$$\alpha_Nv_N\cdot d\theta_i\circ d\pi([\xi_N,v_N])=\alpha_Nd\theta_i\circ d\pi([v_N,[\xi_N,v_N]])=\alpha_Nd\theta_i\circ d\pi(\xi_N)$$

$$\beta_Nv_N\cdot d\theta_i\circ d\pi(\xi_N)=\beta_Nd\theta_i\circ d\pi([v_N,\xi_N])$$

and

$$\tilde{\tau}=O(\delta\ \mathrm{Log}^m\bar{\lambda}+1)-e^{2\delta\sum\theta_is_i}\sum_i\delta s_i\left(d\pi([\xi_N,v_N])\cdot d\theta_i\circ d\pi([\xi_N,v_N])+\right.$$

$$\left.+d\pi(\xi_N)\cdot d\theta_i\circ d\pi(\xi_N)+O(|d\theta_i\circ d\pi|)\ \right)$$

$$=O(\delta\ \mathrm{Log}^m\bar{\lambda}+1)-e^{-2\delta\sum\theta_is_i}\sum\delta s_i\left(d^2\theta_i\left(d\pi([\xi_N,v_N]),d\pi([\xi_N,v_N])\right)\right.$$

$$\left.+d^2\theta\left(d\pi(\xi_N),d\pi(\xi_N)\right)+d\theta_i\left(d\pi([\xi_N,v_N])\cdot d\pi([\xi_N,v_N])+d\pi(\xi_N)\cdot d\pi(\xi_N)\right)+O(|d\theta_i\circ d\pi|)\ \right).$$

It is easy to see that $d\pi([\xi_N,v_N])\cdot d\pi([\xi_N,v_N])$ and $d\pi(\xi_N)\cdot d\pi(\xi_N)$ are bounded independently of N so that

$$\tilde{\tau}=O(\delta\ \mathrm{Log}^m\bar{\lambda}+1)-e^{-2\delta\sum\theta_is_i}\sum_i\delta s_i\left(d^2\theta_i\left(d\pi([\xi_N,v_N]),d\pi([\xi_N,v_N])\right)\right)$$

$$+d^2\theta_i(d\pi(\xi_N, d\pi(\xi_N))) + O(|d\theta_i \circ d\pi|)\ ).$$

From the formula for ξ_N, see Proposition 2, it is clear that $d\pi([\xi_N, v_N])$ and $d\pi(\xi_N)$ have bounded lengths and lengths bounded away from zero and their determinant is bounded away from zero ($\bar\gamma$ is small). Indeed

$$(x^2+y^2)(0)\bar\gamma d\pi(\xi_N) =$$

$$\left(\frac{(x-\bar\gamma y)(0)}{\sqrt{1+\bar\gamma^2}}\cos s_2 + \sqrt{1+\bar\gamma^2}y(0)\sin s_2\right)(e_1-e_2)+$$

$$+\bar\gamma\left((y+\bar\gamma x)(0)\cos s_2 - \sqrt{1+\bar\gamma^2}x(0)\sin s_2\right)e_2$$

$$(x^2+y^2)(0)\bar\gamma d\pi([v_N,\xi_N]) =$$

$$\left(-\frac{(x-\bar\gamma y)(0)}{\sqrt{1+\bar\gamma^2}}\sin s_2 + \sqrt{1+\bar\gamma^2}y(0)\cos s_2\right)(e_1-e_2)-$$

$$-\bar\gamma\left((y+\bar\gamma x)(0)\sin s_2 + \sqrt{1+\bar\gamma^2}x(0)\cos s_2\right)e_2.$$

On the basis $((e_1-e_2), e_2)$, the determinant is:

$$-\bar\gamma\left(\left((y+\bar\gamma x)(0)\sin s_2 + \sqrt{1+\bar\gamma^2}x(0)\cos s_2\right)\left(\frac{(x-\bar\gamma y)(0)}{\sqrt{1+\bar\gamma^2}}\cos s_2 + \sqrt{1+\bar\gamma^2}y(0)\sin s_2\right)+\right.$$

$$\left.+\left((y+\bar\gamma x)(0)\cos s_2 - \sqrt{1+\bar\gamma^2}x(0)\sin s_2\right)\left(-\frac{(x-\bar\gamma y)(0)}{\sqrt{1+\bar\gamma^2}}\sin s_2 + \sqrt{1+\bar\gamma^2}y(0)\cos s_2\right)\right)$$

$$= -\bar\gamma\left((y(0)\sin s_2 + x(0)\cos s_2)^2 + (y(0)\cos s_2 - x(0)\sin s_2)^2\right.$$

$$\left.+\bar\gamma O\left((x^2+y^2)(0)\right)\right) = -\bar\gamma(x^2+y^2)(0)\left(1+O(\bar\gamma)\right).$$

On the other hand, setting $(X = A(e_1-e_2)+Be_2)\|X\|^2 = A^2 + \frac{1}{\bar\gamma^2}B^2$, we have:

$$\left(\frac{(x-\bar\gamma y)(0)\cos s_2}{\sqrt{1+\bar\gamma^2}} + \sqrt{1+\bar\gamma^2}y(0)\sin_2\right) + \left((y+\bar\gamma x)(0)\cos s_2 - \sqrt{1+\bar\gamma^2}x(0)\sin s_2\right)^2 =$$

$$(x(0)^2+y(0)^2)(1+O(\bar\gamma)) = (x^2+y^2)^2(0)\bar\gamma^2\|d\pi(\xi_N)\|^2.$$

The claim follows.

We are ready to prove

Proposition 6. *There exists a constant C independent of $N, \bar{\lambda}$ such that*

$$\tilde{\tau} \leq C$$

Proof. It suffices to build $\theta_i \circ \pi$ (independent of $N, \bar{\lambda}$ etc.) so that

$$d^2\theta_i\left(d\pi(\xi_N), d\pi(\xi_N)\right) + d^2\theta_i(d\pi([v_N, \xi_N], d\pi([v_N, \xi_N])) + O(|d\theta_i \circ d\pi|) \geq 1$$

if $0 \leq \theta_i \leq \frac{1}{2}$. This is possible in view of our claim above.

Furthermore, there exists a constant C_1 independent of N such that

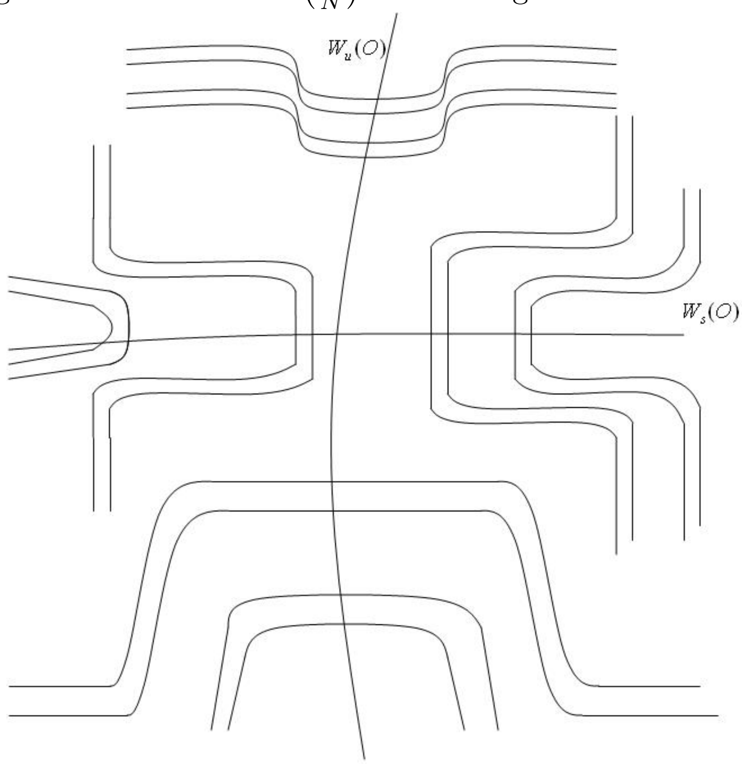
$$d^2\theta_i\left(d\pi(\xi_N), d\pi(\xi_N)\right) + d^2\theta_i(d\pi([v_N, \xi_N]), d\pi([v_N, \xi_N])) + O(|d\theta_i \circ d\pi|) \leq C_1.$$

Finally, we choose δ and $\bar{\lambda}$ so that

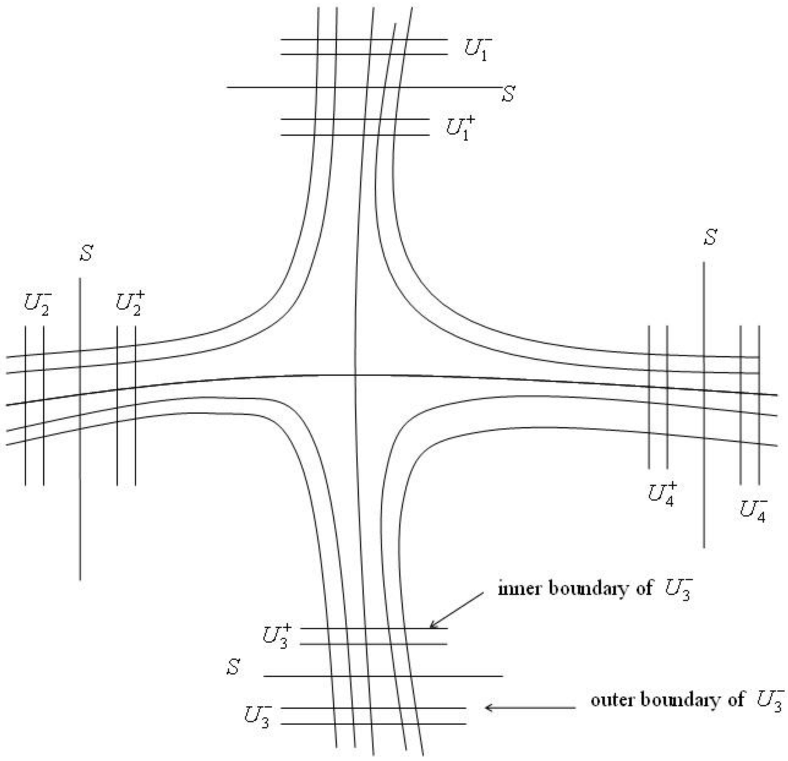
$$O(\delta \operatorname{Log}^m \bar{\lambda} + 1) \leq C_1.$$

The estimate on $\tilde{\tau}$ follows.

Assume now, in a first step, that no periodic orbit generating our homology intersects the stable or the unstable manifold of $\mathcal{O}$. We first complete a diffeomorphism of M and spread the rotation which we have introduced near the attracting and repelling orbit along the stable and unstable manifold of $\mathcal{O}$. We are pointing out, on the drawing below the zones where φ_N is non constant, dropping from 1 to a value $O\left(\frac{1}{N}\right)$ or climbing back to 1.



We create, half-way between each pair of strips, a surface S. We cut then in this picture a thin hyperbolic neighborhood of $W_u(\mathcal{O}) \cup W_s(\mathcal{O})$ and a thinner one.



Between U_i^- and S, $\ker \alpha_N$ turns considerably along $v_N = \varphi_N v$. We can build, with all required bounds on $\frac{\partial^m s_i}{\partial \tau_i^m}$, a function s_i equal to zero on the outer boundary of U_i^- and equal to a large value ℓ_N as we reach S. We can also build $\theta_i = \theta$, a function equal to zero outside of the larger neighborhood of $W_u(\mathcal{O}) \cup W_s(\mathcal{O})$ and equal to 1 on the smaller one. We need here only two functions s_i, s_1 and s_2, s_1 for the repelling orbit and s_2 for the attracting one, with $\theta_1 = \theta_2 = \theta$.

As we reach S, s_1 and s_2 are equal to $\ell_N, \alpha_{N,\lambda} = e^{\delta\theta\ell_N}\alpha$. For $\theta = 1, \alpha = e^{\delta\ell_N}\alpha = \lambda_N\alpha, \lambda_N$ tending to $+\infty$ with N. Thus, our form extends to all of M.

We claim now:

Proposition 7. *Let us consider the periodic orbits of ξ_0 which define the homology at some fixed index k_0 and their unstable manifolds in C_β. The curves on these unstable manifolds do not enter a fixed and small neighborhood of $\mathcal{O}$.*

Proof. All curves x of this type have a tangent vector

$$\dot{x} = a\xi_{N,\lambda} + bv_{N,\lambda}$$

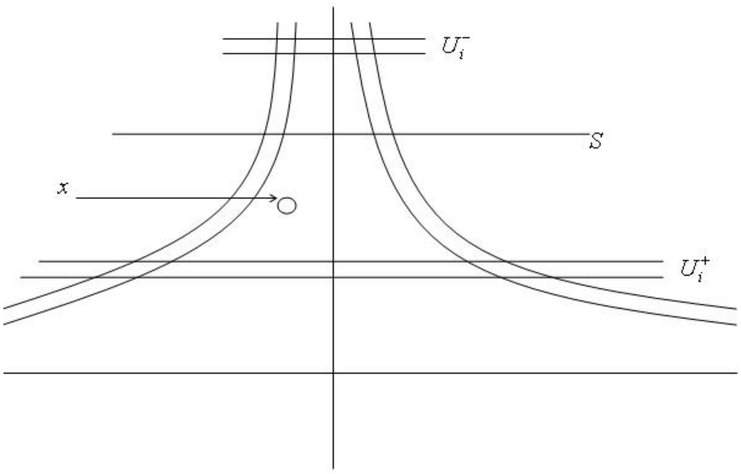
with

$$a \leq a_0; \int_0^1 |b| \leq C$$

a_0 and C are independent of N, a_0 for energy reasons, C because of the bound on $\tilde{\tau}$. Suppose x enters the inner chore. Then, $\dot{x} = O\left(\frac{1}{\lambda_N}\right) + bv_{N,\lambda}$. For N very large, this is basically a piece of orbit of v. If x enters the inner chore from the side i.e. from the boundary of the hyperbolic neighborhood, it stays away from $\mathcal{O}$ since similar orbits of v do not approach $\mathcal{O}$.

On the other hand, if these curves enter the inner chore through the interior boundary of U_i^-, then $v_{N,\lambda} = \frac{\varphi_N v}{\sqrt{\gamma(x)}}$ between this interior boundary and S, i.e. $v_{N,\lambda} = O\left(\frac{1}{N}\right) v$. Since $\int_0^1 |b|$ is bounded and $\xi_N = O\left(\frac{1}{\lambda_N}\right)$, such a curve can hardly move. It cannot enter, assuming it starts in U_i^- or between U_i^- and S, a smaller neighborhood of $\mathcal{O}$ since the piece of orbit of v it spans is so small.

Next, the curve x has no point between U_i^- and S, i.e. lies entirely between S and $\mathcal{O}$. By continuity, we may assume that x starts near S and is therefore entirely contained between S and the outer-boundary of U_i^+, away from the side-boundaries.

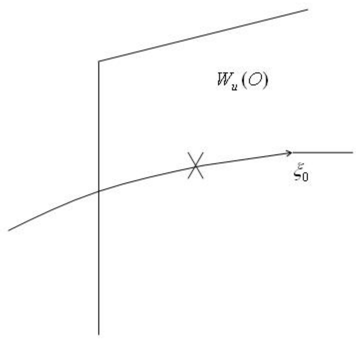


Then,

$$\dot{x} = \frac{\xi_0}{\lambda_N} + bvO\left(\frac{1}{N}\right), \text{ with } \int_0^1 |b| \leq C.$$

In this region, $\alpha_N = \lambda_N \alpha_0, \lambda_N$ a constant and C_β for (α_N, v_N) and (α_0, v) coincide. The curve is tiny and a pseudo-gradient for $\int_0^1 \alpha_0(\dot{x})dt$ on C_{β_0} is a pseudo-gradient for $\int_0^1 \alpha_N(\dot{x})dt$ on C_{β_N}. It is easy to see that such a pseudo-gradient (for $\int_0^1 \alpha_0(\dot{x})dt$ on C_{β_0}) will drive such tiny curves to points locally i.e. keeping away from $\mathcal{O}$. $\square$

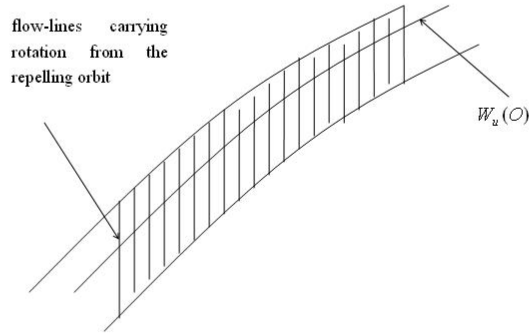
We now have to face the possibility that the periodic orbits of ξ_0 might intersect $W_u(\mathcal{O})$ and $W_s(\mathcal{O})$.



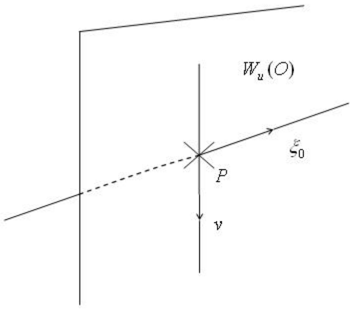
If we try then to carry the rotation from the attractive or repulsive periodic orbit of v to the hyperbolic one, we perturb ξ_0, push away the periodic orbit. If we change the Hamiltonian, we change completely the periodic orbit since we go beyond the effect of a diffeomorphism (carrying rotation is completed through a diffeomorphism once the modification of α_0 into α_N is achieved). If we remove a flow-line neighborhood, as small as we may wish, of the flow-lines of v originating in such periodic orbits (which intersect $W_u(\mathcal{O})$ or $W_s(\mathcal{O})$), we can carry out the rotation of α_N on the complement. How large a neighborhood of the hyperbolic orbit are we carrying then? How much are we missing?

Suppose for example that no periodic of $\xi_)$ intersects, $W_u(\mathcal{O})$, but that several periodic orbits of ξ_0 intersect $W_u(\mathcal{O})$, typically one for simplicity. Then, the rotation from the repelling orbit can be carried out beyond the hyperbolic orbit. These flow-lines (which carry a lot of rotation) fill in a neighborhood of $W_u(\mathcal{O})$ after removing $W_u(\mathcal{O})$.

Using a view from top, we have

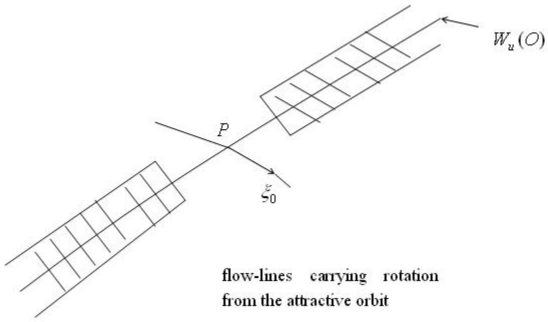


On the other hand, our periodic orbit intersects $W_u(\mathcal{O})$:

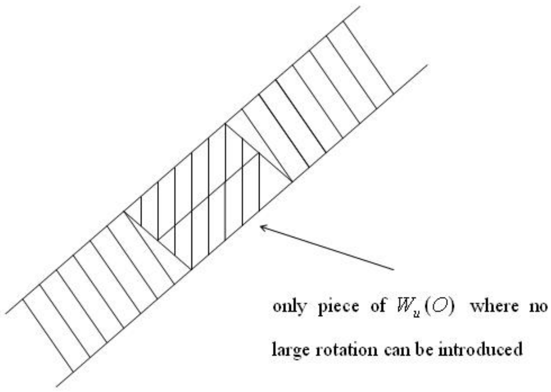


Thus, if we remove a neighborhood of the flow-line of v through P, we can safely bring rotation from the attracting orbit as well.

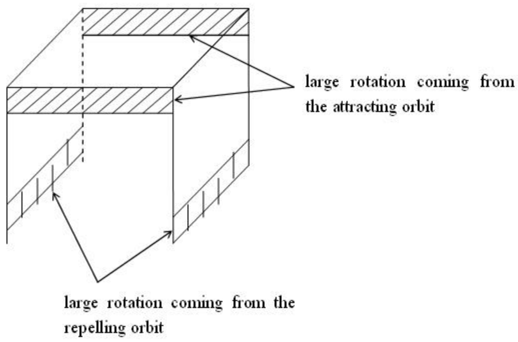
Using a view from top



Thus, below P, we can build a lot of rotation after combining the rotation which we can safely bring from the attractive orbit with the rotation from the repelling orbit



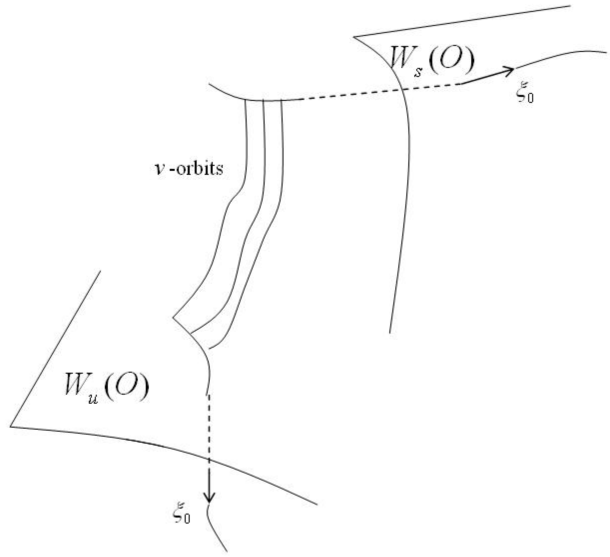
The piece which is left is as small as we wish, we can think of it as a tiny neighborhood of the (downwards) flow-line of v through P. We thus only need to fill this hole. On the boundaries of this hole, we have a lot of rotation distributed as follows



If several periodic orbits intersect $W_u(\mathcal{O})$ and none $W_s(\mathcal{O})$, this hole becomes several holes, but the basic process does not change.

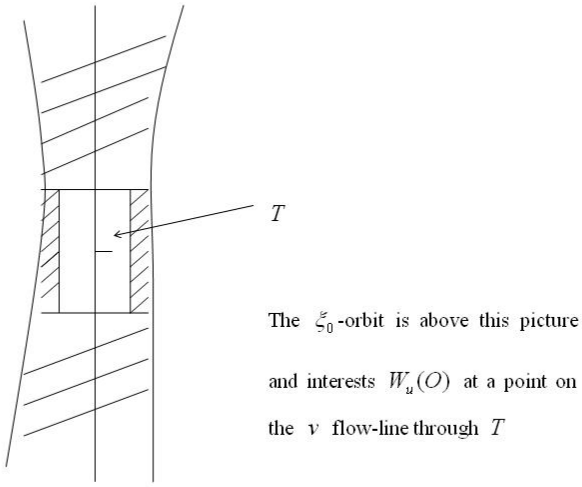
If one (or several) periodic orbit intersects $W_u(\mathcal{O})$ and one (or several) periodic orbit intersect $W_s(\mathcal{O})$, the situation changes since orbits very close to $W_u(\mathcal{O}) \cup W_s(\mathcal{O})$ connect these orbits then.

Any such v-orbit cannot be filled with rotation:



If we remove these flow-lines, we can fill in every remaining flow-line with rotation.

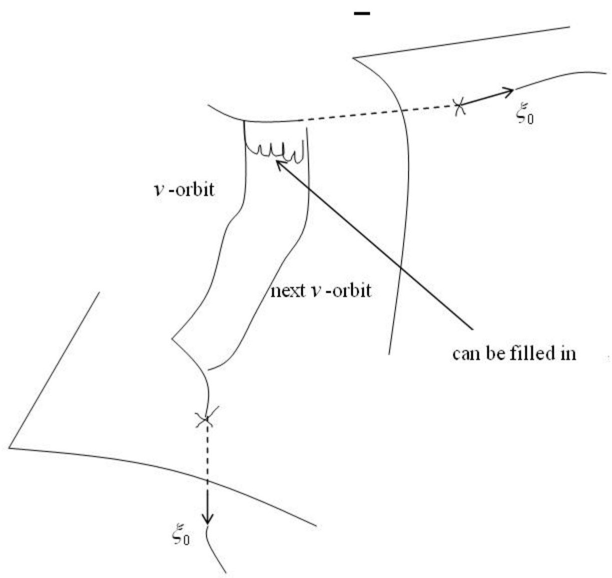
Combining, we can certainly fill with rotation near the intersection point of a periodic orbit with $W_s(\mathcal{O})$ or $W_u(\mathcal{O})$ the following (shaded) set of flow-lines



The ξ_0-orbit is above this picture and intersects $W_u(\mathcal{O})$ at a point on the v-flow-line through T.

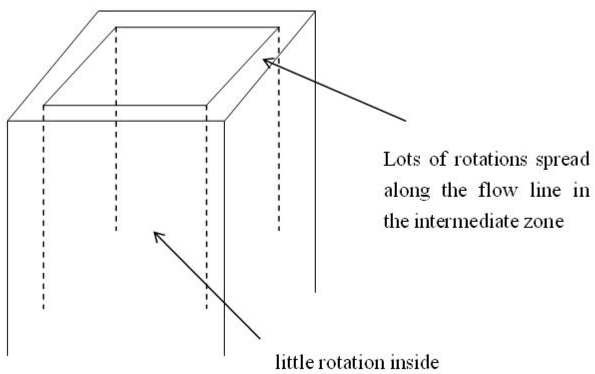
The periodic orbit lies above this picture. We cannot fill the hole more because some v-flow-lines of the hole connect this periodic orbit (which intersects $W_u(\mathcal{O})$ here) with another periodic orbit (intersecting $W_s(\mathcal{O})$ then). We can take this hole to be as small as we wish though after taking thinner neighborhood of the allowed set of flow-lines.

Indeed, any space between the v-orbits connecting the two periodic orbits can be filled in, see the drawing above



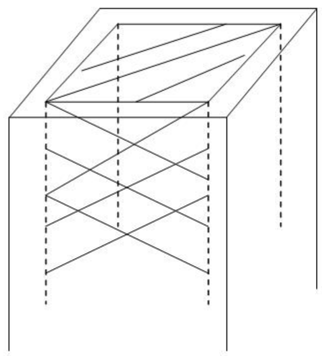
Such spaces are as close as we wish from P. Combining with the rotation brought from above P (repelling or attracting orbit) - carefully removing first the v-flow-lines of the periodic orbit - we derive the "hole-neighborhoods".

Thus, in all the cases studied above, we have derived sets of flow-lines surrounding the hyperbolic periodic orbit and carrying as much rotation as we please but for a finite number of hole-neighborhoods of the following type:



We remove the content of the hole i.e. $\alpha = \alpha_N$ is now defined only at the top of the hole and in the outer neighborhood of flow-lines where there is a lot of rotation.

In the empty hole, we now build a new α_N which rotates considerably before reaching the hyperbolic periodic orbit:



For this purpose, we use (*). We only need to use the appropriate γ and to glue the new rotation so that we have a globally defined α_N, with all required bounds etc.

The first step is to get $\bar{\mu} = 0$ on the boundaries of smaller holes, including the top boundary. Next, we need to rescale the large rotation that we have on each lateral wall so that it stays large but becomes the same all around instead of being split between the top part and the bottom part (see Figure (A)) according to the wall which we are considering.

These two steps are completed in the space between an inner hole and an outer hole (which is smaller than the initial hole, (see Figure (B) below). Then, we can fill the inner hole with a uniform, large rotation.

We first observe that the boundary ∂S, in the flow-box, of the set $S = \{x \in \text{ flow-box}, \varphi(x) \neq 1\}$ is independent of N. It depends only on v, which remains unchanged with N, on the intervals I_0 and on how we carry the rotation below the periodic orbit (coming from the attractive as well as repulsive orbits).

Second, v is transverse to ∂S since v is transverse to the boundaries of the tori where the insertion of a large rotation has taken place.

Third, on each flow-line of v in the box, there are at most three intervals; one where φ is not 1, then one where φ is 1 and a last one where φ is not 1 again.

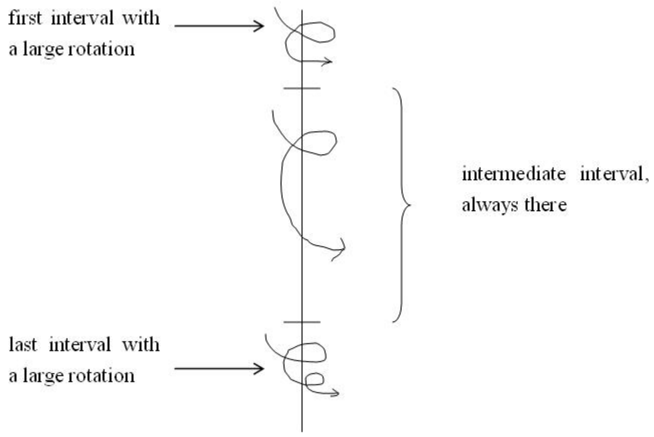
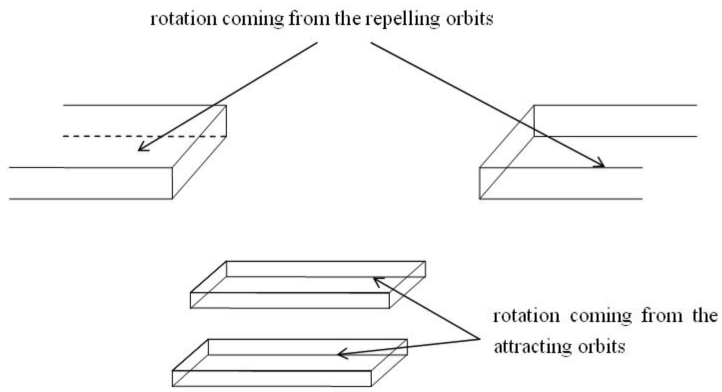
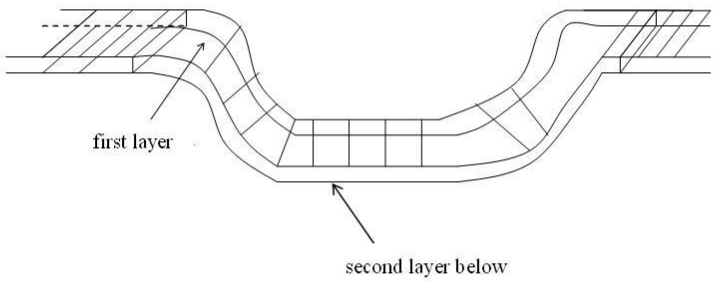


Figure (A)

The only interval which is always present is the one where φ is 1, the "intermediate" one since the large rotations take place in the vicinity of the top and of the bottom of the box. The flow-lines run from top to bottom near (only near) the lateral sides of the box. Each of these flow-lines carries a large rotation which is borrowed either from its top portion or from its bottom one, or from both. Using the large rotations coming from the attractive v-orbit and the repelling v-orbit, we can construct a box around the hole where a lot of rotation is carried around its boundary (all of it)



We draw then the following two layers:



which cut into the upwards rotation, go down to the downwards one and then come back to the upwards one. The top one stays some more upstairs as the lower one speeds up to the lower level to collect rotation from there. If we cut then the central piece and flatten it, we find a thin box

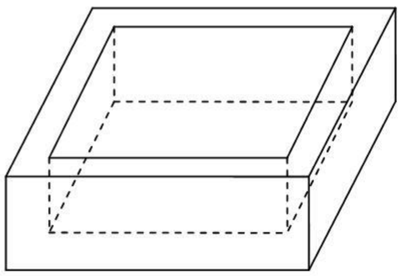


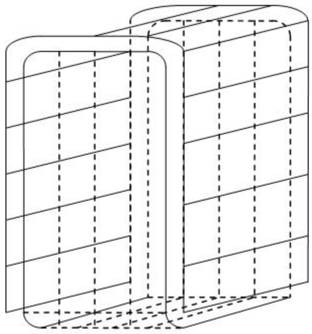
Figure (B)

which carries rotation around its boundary, all around it.

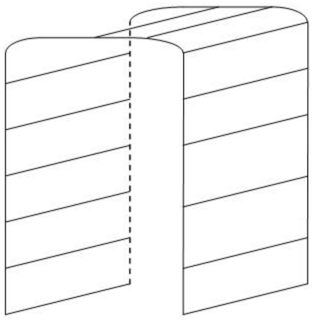
Our arguments apply to this box.

If there is a flow-line (and then several) running from a periodic orbit of $\bar{\xi}$ cutting $W_u(\mathcal{O})$ to a peridic orbit of $\bar{\xi}$ cutting $W_s(\mathcal{O})$, there are two constructions as the one carried out above, one for the top with the periodic orbit cutting $W_u(\mathcal{O})$, the other one for the bottom with the periodic orbit cutting $W_s(\mathcal{O})$. We can match the parts containing a lot of rotation from the top and from the bottom (the boundaries parts). We then fill in partially inside (without matching, leaving a hole which is a neighborhood of the flow-line which connects the two periodic orbits) as if we had only a top or only a bottom.

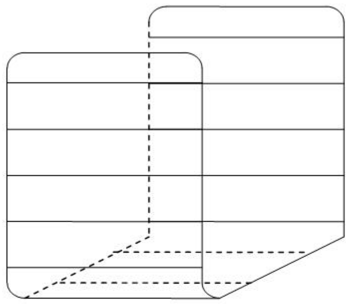
The basic double picture for top and bottom together (not thickened, just flat) is



i.e. there is a top

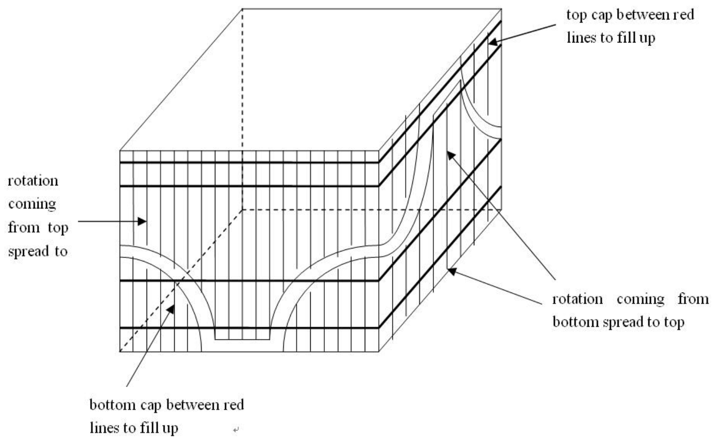


and a bottom



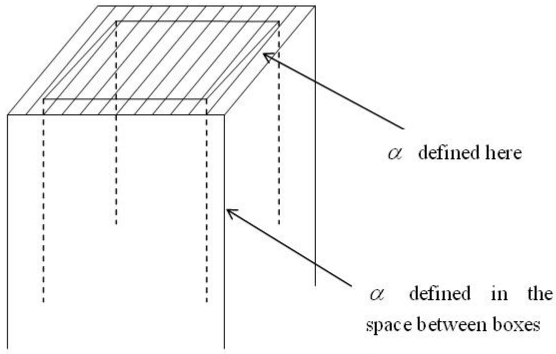
which basically bound the same boundary. Near the boundary, for both of them, there is a lot of rotation. Top and bottom fit together to define a flow-box. The v-flow-line is jailed in the box.

In order to see better how to build our boxes, we draw two ends together and mark with bold lines the two caps of the box from inside which we add rotation, sealing the whole box with rotation all around except for a hole inside it.

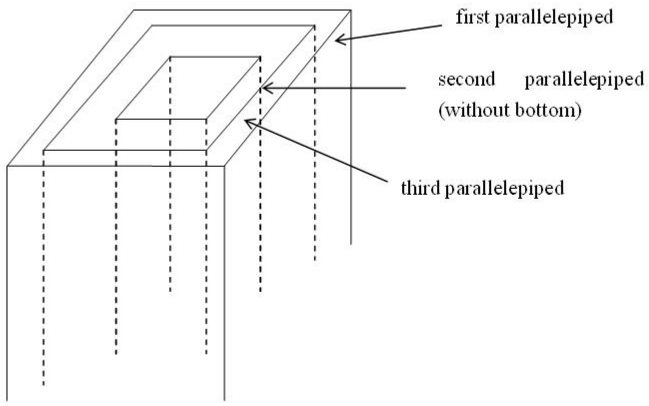


Observation Between the two bold lines defining either of the top or the bottom caps, the v-flow-lines carry a lot of rotation near the boundary. This fact is used to extend the rotation inside the box, sealing it off; a hole is left inside.

We now have our initial flow-box and an inner, smaller one without bottom. α is defined in the space between the two boxes and or the top side.



The only space where ξ and α are not defined is a parallelepiped with a top and with a bottom. We build another yet smaller parallelepiped



Since ∂S is independent of N, we can easily extend the function φ (the parametrization along v) between parallelepiped 2 and 3 and extend as well $\bar{\mu}, \xi$ etc. The uniformity of ∂S allows us to keep all bounds. $\bar{\mu}$ (extended using the function γ and (*), ξ, α are extended using (*)) is kept equal to zero on the extension of S (which we may complete as we please between box 2 and box 3 as long as it matches with the boundary data - of ∂S- on the boundaries - top and lateral - of box 2).

Such an extension of ξ, α etc. enjoys the same bounds. Indeed, outside of S, ξ was $\bar{\xi}$. Thus, on $\partial S, \xi = \bar{\xi}, [\varphi v, \xi] = [v, \bar{\xi}]$, the v- interval outside of S is "large" as pointed out (independent of N) so that (*) provides C^∞-bounds depending only on γ. τ is therefore bounded outside of S. Inside S, it is bounded because $\bar{\mu} = 0$, thus $\tau_v (= -\bar{\mu}_{\xi\xi} - \tau\bar{\mu})$ is zero and τ equals the value it has on the boundaries of the "intermediate, large" interval.

One issue to worry about is the glueing of the data of ξ, $[\varphi v, \xi]$ derived from the initial conditions near the top of the box after the use of (*) once we reach the bottom part of the box.

Since box 2 has no bottom side, we can sidestep this problem here, but one can easily overcome it manipulating (*) above the boxes.

We may assume now that (the extension of $\bar{\mu}$ into γ is as we please, subject to $\gamma = 0$ on the extension of S) γ is zero identically on the boundaries of box 3, including the top side .

One can get $\bar{\mu} = 0$ on the top side after a modification of $\bar{\mu}$ into zero, using (*) and γ, along small flow-lines originating in this top part - Curving then the top part, we can build a flow-box 1 which has the same lateral boundaries than the former flow-box 1 and still carries a large rotation on all flow-lines between box 1 and box 2, while $\bar{\mu} = 0$ on the top portion of box 3.

We then observe that the rotation of γ on the lateral boundaries of box 3 is large, either on the top portion or on the bottom one. This is embedded in the construction and is due to the fact that $\int \frac{dx}{\varphi(x)}$ (φ is the parametrization along v) is large on each of these flow-lines. We then extend φ so that it becomes constant (small obviously) on all the lateral sides of a yet smaller parallelepiped box 4. From there, the extension inside box 4 is immediate.

We need to check that this last modification, the spreading of the rotation so that it becomes uniform, keeps all bounds holding true. This rescaling is typically derived through the diffeomorphism of [0,1].

$$\begin{aligned} [0,1] &\longrightarrow [0,1] \\ x &\longrightarrow \frac{t\int_0^x \frac{ds}{\varphi(s)} + (1-t)x}{\int_0^1 \frac{ds}{\varphi(s)}} \end{aligned}$$

$[0,1]$ is the time along the v-flow-line from bottom to top, φ is the function built with the rotations, as such it depends on the base point z of the flow-line. $t = t(z)$ depends also on the base point of the flow-line z, which is on the top of the box. t is zero on the lateral boundary of box 2 and 1 on the lateral boundary of box 3.

This gives rise to a diffeomorphism $\gamma_{s(y)}(y)$ of the space between box 2 and box 3. γ_s is the one-parameter of v.

$D\gamma_s$ is of course bounded. ds is the differential of

$$\frac{t(z)\int_0^x \frac{ds}{\varphi(s)} + (1-t(z))x}{\int_0^1 \frac{ds}{\varphi(s)}} - x.$$

Observe that $\frac{1}{\varphi}$ is at most CN, hence is upperbounded by $C_1 \int_0^1 \frac{ds}{\varphi(s)}$, since we may assume that the total rotation of these flow-lines, between box 2 and box 3, is at least $c_0 N$.

We also claim that

$$\left| \int_0^x \frac{\partial \varphi}{\partial z} \frac{ds}{\varphi^2} \right| \leq C_1 \int_0^1 \frac{ds}{\varphi(s)}.$$

Indeed, the top of the flow-box is transverse to v. φ is a function of s and as such, between the two tori, does not depend on the flow-lines. The dependency on the flow-lines is due to the transformation (given, independent of N) which brings the rotation to the flow-box. Thus,

$$\left|\frac{\partial\varphi}{\partial z}\right| \leq C|\varphi'(s)|$$

and

$$\left|\int_0^x \frac{\partial\varphi}{\partial z}\frac{ds}{\varphi^2}\right| \leq C\int_0^x \frac{|\varphi'|}{\varphi^2}ds \leq C' \text{ Max } \frac{1}{\varphi}.$$

Thus, again,

$$\frac{\left|\int_0^x \frac{\partial\varphi}{\partial z}\frac{ds}{\varphi^2}\right|}{\int_0^1 \frac{ds}{\varphi(s)}} \leq C_1$$

dt is also bounded independently of N. Thus, ds is bounded independently of N.

We thus have ($\tilde{\varphi}$ is the parametrization which we built).

Proposition 8. *$\xi, [\tilde{\varphi}v, \xi], \bar{\mu}, \tau$ are bounded. Furthermore,*

$$\xi\cdot\xi, [\tilde{\varphi}v,\xi]\cdot[\tilde{\varphi}v,\xi], \xi\cdot[\tilde{\varphi}v,\xi], [\tilde{\varphi}v,\xi]\cdot\xi$$

are bounded independently of N transversally to v.

Proof. Since $D\gamma_{s(y)} + ds(\cdot)v$ is bounded, $\xi, [\tilde{\varphi}v, \xi]$ are bounded. $\bar{\mu}$ and τ are bounded by construction.

$\xi\cdot\xi$ etc are initially bounded but $\gamma_{s(y)}(y)$ could have an effect. However, denoting $\tilde{\xi}$ the initial ξ,

$$\xi = D\gamma_{s(y)}(\tilde{\xi}) + ds(\tilde{\xi})v = D(\gamma_{s(y)})(\tilde{\xi}).$$

Any further derivative taken on ξ through a vector-field which reads

$$D\gamma_{s(y)}(X) + ds(X)v = D(\gamma_{s(y)})(X)$$

would yield derivatives of $\tilde{\xi}$ along X which are bounded transversally to v (and v is mapped onto θv by $D(\gamma_{s(y)})$) if X splits on $\tilde{\xi}, [\varphi v, \tilde{\xi}]$, i.e. if $D\gamma_{s(y)}(X) + ds(X)v$ splits on $\xi, [\tilde{\varphi}v, \xi]$, derivatives of v, which are bounded and would yield derivatives of $D(\gamma_{s(y)})$ which might be unbounded.

But

$$D(\gamma_{s(y)}) = D\gamma_{s(y)} + dsv.$$

Derivatives fo $D\gamma_{s(y)}$ are bounded as well as derivatives of v. Derivatives of ds might be unbounded; but these are multiplied by v and our estimate is transversal to v.

Our construction of the functions s_1, s_2 etc proceeds then as in the case when no periodic orbit of the contact vector-field intersected $W_u(\mathcal{O})$ and $W_s(\mathcal{O})$. Proposition 8 holds.

Lemma 12. *The function x^2+y^2 has no local maximum near the repelling or attracting orbits along the trajectories of $\bar{\xi}$.*

Proof. We recall that $\bar{\xi} = \frac{\partial}{\partial x} = \bar{\gamma}\frac{\partial}{\partial y}$ so that

$$\bar{\xi}\cdot\bar{\xi}\cdot(x^2+y^2) = 2\bar{\xi}\cdot(x-\bar{\gamma}y) = 2(1+\bar{\gamma}^2) > 0.$$

$\square$

Corollary 4. *There is no "small" $\bar{\xi}$-trajectory exiting from T_2 and coming back after a short time.*

Next, we establish the following qualitive result

Lemma 13. *For $\bar{\gamma}$ small enough, τ_N is negative in $T_2 - T_1$.*

Proof. We write

$$\xi_N = (A\cos s_1 + B\sin s_1)e_2 + (A_1\cos s_1 + B_1\sin s_1)\varphi v + De_1$$

φv is $\frac{\partial}{\partial s_1}, e_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, e_1 = 20\frac{\partial}{\partial\theta}$.

We know that

$$[\xi_N,[\varphi v,\xi_N]] = \tau_N\varphi v$$

i.e.

$$[\xi_N,\frac{\partial\xi_N}{\partial s_1}] = \tau_N\frac{\partial}{\partial s_1}.$$

Also

$$\frac{\partial e_1}{\partial s_1} = \frac{\partial e_2}{\partial s_1} = 0.$$

Since $ds_1(e_1) = ds_1(e_2) = 0$ and since $\frac{\partial}{\partial\theta}A_1 = \frac{\partial}{\partial\theta}B_1 = 0, e_1$ has no contribution in $[\xi_N,\frac{\partial\xi_N}{\partial s_1}]$ and

$$[\xi_N,\frac{\partial\xi_N}{\partial s_1}] = \Big[(A\cos s_1 + B\sin s_1)e_2 + (A_1\cos s_1 + B_1\sin s_1)\frac{\partial}{\partial s_1},$$

$$(-A\sin s_1 + B\cos s_1)e_2 + (-A_1\sin s_1 + B_1\cos s_1)\frac{\partial}{\partial s_1}\Big] =$$

$$= \frac{\partial}{\partial s_1}\big(-(A_1\cos s_1 + B_1\sin s_1)^2 - (B_1\cos s_1 - A_1\sin s_1)^2 +$$

$$+(A\cos s_1 + B\sin_1)(-e_2\cdot A_1\sin s_1 + e_2\cdot B_1\cos s_1) -$$

$$-(-A\sin s_1 + B\cos s_1)(e_2\cdot A_1\cos_1 + e_2\cdot B_1\sin s_1)\big)$$

$$= \left(-(A_1^2 + B_1^2) + Ae_2 \cdot B_1 - Be_2 \cdot A_1\right) \frac{\partial}{\partial s_1}.$$

Thus,

$$\tau_N = -(A_1^2 + B_1^2) + Ae_2 \cdot B_1 - Be_2 \cdot A_1.$$

From Proposition 2, we derive

$$\begin{aligned}
A &= \frac{-(x - \bar\gamma y)}{\bar\gamma (x^2 + y^2)\sqrt{1 + \bar\gamma^2}}(0) + \frac{(y + \bar\gamma x)(0)}{(x^2 + y^2)(0)} \\
B &= \frac{-y(0)\sqrt{1 + \bar\gamma^2}}{\bar\gamma (x^2 + y^2)(0)} - \frac{\sqrt{1 + \bar\gamma^2}x(0)}{(x^2 + y^2)(0)}. \\
A_1 &= -\frac{(x - \bar\gamma y)(0)}{\bar\gamma (x^2 + y^2)(0)\sqrt{1 + \bar\gamma^2}} \\
B_1 &= -\frac{y(0)\sqrt{1 + \bar\gamma^2}}{\bar\gamma (x^2 + y^2)(0)} - 2\frac{(x - \bar\gamma y)(0)}{(x^2 + y^2)(0)\sqrt{1 + \bar\gamma^2}}.
\end{aligned}$$

Observe that

$$\begin{aligned}
e_2 \cdot (x^2 + y^2)(0) &= 0 \\
\bar\gamma (x^2 + y^2)(0)A &= -x(0) + O(\bar\gamma)(|x| + |y|) \\
\bar\gamma (x^2 + y^2)(0)B &= -y(0) + O(\bar\gamma)(|x| + |y|) \\
\bar\gamma (x^2 + y^2)(0)A_1 &= -x(0) + O(\bar\gamma)(|x| + |y|) \\
\bar\gamma (x^2 + y^2)(0)B_1 &= -y(0) + O(\bar\gamma)(|x| + |y|) \\
e_2 \cdot (\bar\gamma (x^2 + y^2)(0)A_1) &= -y(0) + O(\bar\gamma)(|x| + |y|) \\
e_2 \cdot (\bar\gamma (x^2 + y^2)(0)B_1) &= x(0) + O(\bar\gamma)(|x| + |y|).
\end{aligned}$$

Thus,

$$\tau_N = \frac{1}{\bar\gamma (x^2 + y^2)(0)^2}(-2(x(0)^2 + y(0)^2)(1 + O(\bar\gamma^2))) = -2\frac{(1 + O(\bar\gamma^2))}{\bar\gamma^2 (x^2 + y^2)(0)} < 0.$$

□

Lemma 14. *If all the θ_i's involved in the construction of $\tilde\tau$ are small at a point x of M ($\theta_i \leq c, c$ independent of N), then*

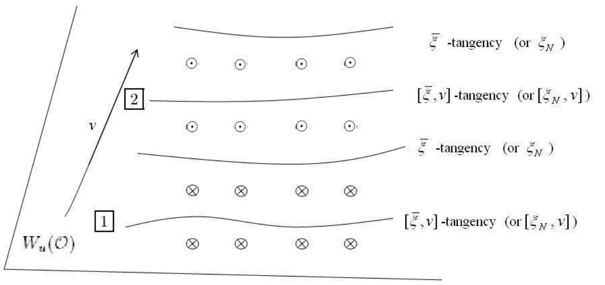
$$\tilde\tau \leq \frac{-2}{\bar\gamma^2 (x^2 + y^2)(0)}(1 + O(\bar\gamma^2))e^{-\delta \sum \theta_i s_i} + O(\delta \ Log^m \bar\lambda).$$

Proof. Coming back to $\tilde{\tau}$, we recognize that the term $O(1)$ in its expression comes from $\frac{B\tau}{\nu(x)\lambda}$ and that, for θ_i small,

$$[\xi_N, v_N] \cdot (d\theta_i \circ d\pi)([\xi_N, v_N]) + \xi_N \cdot (d\theta_i \circ d\pi)(\xi_N)$$

is positive. The claim follows.

$W_s(\mathcal{O})$ and $W_u(\mathcal{O})$ are tangent to v. Let us for example focus here on $W_u(\mathcal{O})$. Along v, as we move away from $\mathcal{O}$ to go to the attracting orbit of $v, \bar{\xi}, \xi_N$ rotate as well as $[\bar{\xi}, v]$ and $[\xi_N, v_N]$. This builds a sequence of lines (closed lines) of tangency of $\bar{\xi}(\xi_N), [\bar{\xi}, v]([\xi_N, v_N])$ to $W_u(\mathcal{O})$



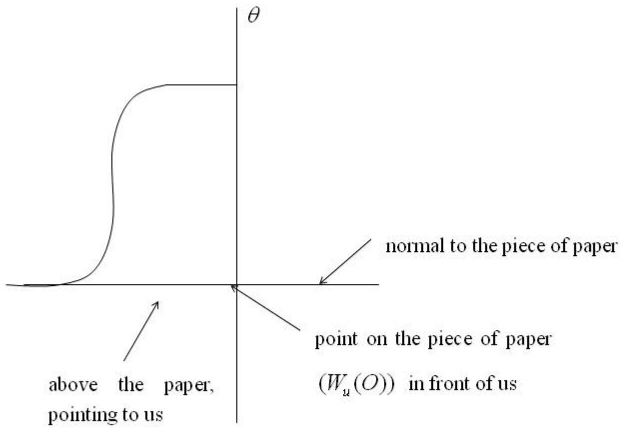
These lines are sizably spaced along v for $\bar{\xi}, [\bar{\xi}, v]$, along v_N for $\xi_N, [\xi_N, v_N]$.

$\otimes$ designates the region where $\bar{\xi}$ (or ξ_N) points into the paper across $W_u(\mathcal{O})$ while $\odot$ designates the region where $\bar{\xi}$ (or ξ_N) points towards us.

Let us consider the vector-field

$$-s\delta d\theta([\xi, v])v$$

where $\xi = \xi_{N,\lambda}$ and $v = v_{N,\lambda}$. θ here is zero above the piece of paper and builds up to 1 as we approach it



Lemma 15. *Between* $\boxed{1}$ *and* $\boxed{2}$, $-s\delta d\theta([\xi, v])v$ *points downwards from* $\boxed{2}$ *and* $\boxed{1}$. *Below* $\boxed{1}$ *until the next* $\bar{\xi}$*-tangency line,* $-s\delta d\theta([\xi, v])v$ *points upwards towards* $\boxed{1}$.

Proof. Observe that, since $d\theta(v) = 0$,

$$v \cdot d\theta(\xi) = d\theta([v, \xi]).$$

From $\boxed{1}$ to $\boxed{2}$ along a v-flow-line, $d\theta(\xi)$ decreases (it is positive near $\boxed{1}$, negative near $\boxed{2}$) so that

$$d\theta([v, \xi]) < 0, d\theta([\xi, v]) > 0 \text{ between } \boxed{1} \text{ and } \boxed{2}.$$

The claim follows.

We now have

Lemma 16. *Assume that* $\delta s \geq M$. *There exists* $c(M) > 0, c(M)$ *tending to zero with* $\frac{1}{M}$ *and* δ, *such that any piece of* $\xi_{N,\lambda}$*-orbit entering and exiting the region of modification between* $\boxed{1}$ *and* $\boxed{2}$ *stays in a* $c(M)$*-neighborhood of the* $\bar{\xi}(\xi_N)$ *line of tangency.*

Proof. We take local coordinates between $\boxed{1}$ and $\boxed{2}$ where $W_u(\mathcal{O})$ is $y = 0, v$ is $\frac{\partial}{\partial z}$ $\left(v_N = \frac{\partial}{\partial z}\right)$ and $\frac{\partial}{\partial x}$ is tangent to the $\bar{\xi}$ or ξ_N tangency line. $\xi_{N,\lambda}$ reads up to the factor $\frac{1}{\lambda}$ as

$$\bar{\xi} - s\delta d\theta([\xi, v])v + O(\delta\theta)[\bar{\xi}, v]$$

$\bar{\xi}$ has near the tangency line a non-zero component on $\frac{\partial}{\partial x}$. It thus reads as

$$\begin{cases} \theta_0 + a_1(x - x_0) + b_1 y + c_1 z + \text{ higher order} \\ -z\gamma(x, y, z) \qquad , \text{ with } \theta_0 \neq 0, \gamma(x_0, 0, 0) \neq 0, \text{ positive} \\ \mu_0 + a_2(x - x_0) + b_2 y + c_2 z + \text{ higher order.} \end{cases}$$

near $(x_0, 0, 0)$. The y-axis goes from left to right through $W_u(\mathcal{O})$. Thus $\xi_{N,\lambda}$ reads after setting $\theta = \bar{M}(y+\eta)^{+4}$ for example

$$O(\bar{M}(y+\eta)^{+4}\delta) + \begin{cases} \theta_0 + a_1(x-x_0) + b_1 y + c_1 z + \text{ higher order} \\ -z\gamma(x,y,z) \\ \mu_0 + a_2(x-x_0) + b_2 y + c_2 z - \delta s 4\bar{M}(y+\eta)^{+3}\bar{c}(x,y,z) + \text{ higher order.} \end{cases}$$

with $\bar{c}(x_0, 0, 0) > 0$ and $\bar{M} = \bar{M}(x,y,z)$ very-large.

This provides the general form (up to meaningless details) of $\xi_{N,\lambda}$ near the line of tangency of $\bar{\xi}$. The size of the neighborhood where this form holds does not depend on $\bar{M}, M$. The higher terms are independent on $M, \bar{M}$.

Observe that if $(\bar{c}_0 \leq \bar{c})$

$$4M\bar{M}(y+\eta)^3\bar{c}_0 \geq C_1,$$

C_1 a fixed constant, then

$$\dot{z} < 0.$$

Thus, if at such point $z < O(\bar{M}(y+\eta)^{+4}\delta), \dot{y} = -z\gamma + O(\bar{M}(y+\eta)^{+4}\delta) > 0$. $4M\bar{M}(y+\eta)^3\bar{c}_0$ is larger than C_1 thereafter, z remains less than$O(\bar{M}(y+\eta)^{+4}\delta)$, the $\xi_{N,\lambda}$ piece of orbit cannot exit without crossing the chore.

Thus, we need

$$4M\bar{M}(y+\eta)^3\bar{c}_0 \leq C_1 \text{ as long as } z < -c_2\bar{M}\delta(y+\eta)^{+4}.$$

Assume now that

$$z(0) \leq -c(M), y + \eta(0) \geq 0.$$

Then,

$$z(t) \leq -c(M) + \bar{c}\Delta t \quad , \bar{c} \text{ independent of } M, \bar{M}.$$

Thus,

$$z\left(\frac{c(M)}{2\bar{c}}\right) \leq -\frac{c(M)}{2}$$

and for $0 \leq t \leq \frac{c(M)}{2\bar{c}}$, taking $\bar{M}\delta < 1$:

$$y + \eta(t) \geq \frac{c(M)}{4}\underline{\gamma}t.$$

Thus,

$$\begin{cases} (y+\eta)\left(\frac{c(M)}{2\bar{c}}\right) \geq \frac{c(M)^2\underline{\gamma}}{8\bar{c}} \\ z\left(\frac{c(M)}{2\bar{c}}\right) \leq -\frac{c(M)}{2}. \end{cases}$$

It suffices then to take

$$\begin{cases} c(M) > 2c_2\bar{M}\delta \\ 4M\bar{M}\bar{c}_0\frac{c(M)^6\gamma^3}{\gamma^3\bar{c}^3} = 2C_1. \end{cases}$$

and we have a contradiction.

We thus need to have

$$0 \geq z(0) \geq -c(M)$$

at the time of entry.

We now follow the piece of orbit of $\xi_{N,\lambda}$. As z reaches the value $\frac{\bar{c}(M)}{2}-$ if it does; if it does not, we are done - either $\dot{z} < 0$ and z becomes less than $\frac{\bar{c}(M)}{2}$ or $\dot{z}$ is non negative. This forces ($\dot{\overline{s\delta}} = \delta$ so that $s\delta = s\bar{\delta} + 0(\delta) = s\bar{\delta}(1 + o(1))$)

$$(y+\eta)^+ \leq \left(\frac{C_1}{4s\bar{\delta}\bar{M}\bar{c}_0}\right)^{1/3}.$$

As long as z remains larger than $\frac{\bar{c}(M)}{4}$,

$$\dot{\overline{(y+\eta)^+}} \leq -\frac{\bar{c}(M)}{8}$$

and

$$0 \leq (y+\eta)^+ \leq -\frac{\bar{c}(M)}{8}\Delta t + \left(\frac{C_1}{4s\bar{\delta}\bar{M}\bar{c}_0}\right)^{1/3}. \tag{32}$$

This forces

$$\Delta t < \frac{8}{\bar{c}(M)}\left(\frac{C_1}{4s\bar{\delta}\bar{M}\bar{c}_0}\right)^{1/3} = \Delta t_{\ \max}.$$

Assume that $z(t_0) = \frac{\bar{c}(M)}{2}$ and for $t \in [t_0, t_1]$,

$$z(t) \geq \frac{\bar{c}(M)}{4}.$$

Then,

$$(y+\eta)^+ \leq \left(\frac{C_1}{4s\bar{\delta}\bar{M}\bar{c}_0}\right)^{1/3} \text{ for } t \in [t_0, t_1]$$

and

$$\dot{z}(t) \geq -C_2$$

so that

$$z(t) \geq z(t_0) - C_2(t - t_0) = \frac{c(M)}{2} - c_2(t - t_0).$$

$\frac{\bar{c}(M)}{2} - c_2(t - t_0)$ is larger than $\frac{\bar{c}(M)}{4}$ if

$$t - t_0 \leq \frac{\bar{c}(M)}{2c_2}.$$

We thus know that until $\frac{\bar{c}(M)}{2c_2}$, $z(t)$ is larger than $\frac{\bar{c}(M)}{4}$. We thus ask that

$$\Delta t_{\max} = \frac{8}{\bar{c}(M)} \left(\frac{C_1}{4s\bar{\delta}\bar{M}\bar{c}_0} \right)^{1/3} \leq \frac{\bar{c}(M)}{2c_2}$$

e.g.

$$\bar{c}(M) \sim c_3 \left(\frac{1}{M\bar{M}} \right)^{1/6}.$$

Then (32) holds until the time of exit i.e.

$$t_1 - t_0 \leq \frac{8}{\bar{c}(M)} \left(\frac{C_1}{4s\bar{\delta}\bar{M}\bar{c}_0} \right)^{1/3}$$

and

$$z(t_1) \leq z(t_0) + C(t_1 - t_0) = \frac{\bar{c}(M)}{2} + \left(\frac{\tilde{C}}{\overline{s\bar{\delta}\bar{M}}} \right)^{1/6} \leq K\bar{c}(M).$$

□

Appendix 1

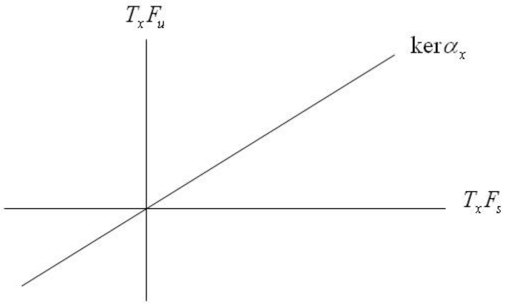
The normal form for (α, v) when α does not turn well.

We consider a hyperbolic orbit $\mathcal{O}$ of v. We establish:

Proposition 9. *There is, up to diffeomorphism, a unique local model for (α, v) around $\mathcal{O}$ such that α does not turn well.*

Proof. Let σ be a section to v at $x_0 \in \mathcal{O}$. Since $\mathcal{O}$ is hyperbolic, it has a stable and an unstable manifold; they can be seen as two foliations $\mathcal{F}_u, \mathcal{F}_s$ with traces A_u, A_s in σ.

Since α does not turn well along v, $\ker \alpha$ along $\mathcal{O}$ is never tangent to $\mathcal{F}_u$, neither is it tangent to $\mathcal{F}_s$. Otherwise the mononicity of the rotation of $\ker \alpha$ would imply an infinite amount of rotation. Thus, $\ker \alpha$ is contained between $T\mathcal{F}_u$ and $T\mathcal{F}_s$, it lies in exactly one of the sectors defined by the tangent spaces along $\mathcal{O}$ to $\mathcal{F}_u$ and $\mathcal{F}_s$.



This property extends, by continuity, to a small neighborhod of $\mathcal{O}$. Let σ be a small section to v at x_0 and let ℓ be the Poincaré return map. Along the v-orbit from $x \in \sigma$ to $\ell(x)$, $\ker\alpha_y$ rotates monotonically with respect to the tangent spaces to the foliations $T_y\mathcal{F}_u$ and $T_y\mathcal{F}_s$, since these tangent spaces are transported by v. If we declare $T_x\mathcal{F}_u$ and $T_x\mathcal{F}_s$ to be orthogonal to each other, we have

$$0 < \theta(x) < \frac{\pi}{2}$$

$\theta(x)$ designates the amount of rotation of $\ker\alpha_y$ from x to $\ell(x)$. Given two distinct contact structures, both having v in their kernel and both not turning well along v, we have two functions $\theta_1(x), \theta_2(x)$, both between 0 and $\frac{\pi}{2}$.

We complete around $\mathcal{O}$ a rotation which maps $\ker\alpha_1$ to $\ker\alpha_2$ at x_0. Along the rotation, $\ker\alpha_1$ remains in the same quadrant for all x in a small neighborhood of $\mathcal{O}$. It is then easy to scale the speed of the rotation of $\ker\alpha_1$ so that it coincides along $\mathcal{O}$ with $\ker\alpha_2$. This rescaling is a diffeomorphism of a neighborhood of $\mathcal{O}$ which may be achieved through a reparametrization of the v-orbits.

$\ker\alpha_1$ and $\ker\alpha_2$ are now close in a whole neighborhood of $\mathcal{O}$ and they both have v in their kernel.

Since both planes rotate monotonicaly, are very close and have the same limits at infinity (due to the hyperbolic behavior of v), it is possible to bring one onto the other through a reparametrization of each v-orbit. □

The normal form of (α, v) near an attractive periodic orbit of v.

Let $\mathcal{O}$ be a periodic orbit of v. We establish:

Proposition 10. *There are suitable coordinates (θ_1, x, y), θ_1 being an angular coordinate along $\mathcal{O}$ such that α reads $\lambda(\theta_1, x, y)(xd\theta_1 + dy)$.*

Proof. $\ker\alpha$ is tangent to $\mathcal{O}$ and there is an additional direction along $\mathcal{O}$ defining $\ker\alpha$. v and this additional direction define an orientable frame in $\ker\alpha$. If we add to these two vectors the contact vector-field ξ of α, we build a frame for M^3 along $\mathcal{O}$. We thus can take, along $\mathcal{O}$, this additional direction to be $\frac{\partial}{\partial x}$ and ξ to be $\frac{\partial}{\partial y}$, $v=\frac{\partial}{\partial \theta}$.

α then takes the form:

$$\big(a(\theta)x+b(\theta)y+O(x^2+y^2)\big)d\theta+\big(1+a_2(\theta)x+b_2(\theta)y+O(x^2+y^2)\big)dy+ \\ +\big(a_1(\theta)x+b_1(\theta)y+O(x^2+y^2)\big)dx.$$

Using Gray's theorem, its use leaves $\mathcal{O}$ unchanged as can be checked, we can get rid of all second order terms after the introduction of a related diffeomorphism.

Rescaling, α reads (up to a multificative factor):

$$\big(a(\theta)x+b(\theta)y\big)d\theta+\big(a_1(\theta)x+b_1(\theta)y\big)dx+dy.$$

Since $\alpha\wedge d\alpha$ is a volume form, $a(\theta)$ is non-zero for every $\theta\in[0,2\pi]$.

We rewrite α (rescaled) as:

$$\big(a(\theta)x+b(\theta)y\big)d\theta+\big(1-b_1(\theta)x\big)dy+d\big(a_1(\theta)\frac{x^2}{2}+b_1(\theta)xy\big)-\frac{x^2}{2}a_1'(\theta)d\theta-b_1'(\theta)xyd\theta.$$

We remove as above $\frac{x^2}{2}\ a_1'(\theta)d\theta-b_1'(\theta)xy\ d\theta$ since it is second order. $d(a_1(\theta)\frac{x^2}{2}+b_1(\theta)xy)$ is closed and $o(1)$. We may also remove it using again Gray's theorem.

We are left, after rescaling with:

$$\big(a(\theta)x+b(\theta)y\big)d\theta+dy \quad \text{with} \quad a(\theta)\neq 0.$$

Setting

$$d\theta_1=a(\theta)d\theta,$$

we find

$$\big(x+\tilde{b}_1(\theta)y\big)d\theta_1+dy.$$

The family

$$\big(x+t\tilde{b}_1(\vartheta)y\big)d\vartheta_1\, dy$$

is a one-parameter family of contact forms. For all of them, there is a vector-field v_t in their kernel having $\mathcal{O}$ as a periodic orbit,. The family is constant equal to dy on $\mathcal{O}$.

We thus can use Gray's theorem and reduce up to rescaling, α to

$$xd\theta_1+dy$$

as claimed.

Let us consider two vector-fields v_1,v_2 having $\mathcal{O}$ as a periodic orbit. Up to reparameterization, they read:

$$\frac{\partial}{\partial\theta_1}-x\frac{\partial}{\partial y}+\delta x_i\frac{\partial}{\partial x}.$$

Assuming that $\mathcal{O}$ is attractive for both of them, we can find functions $a_1(\theta), a_2(\theta), b_1(\theta), b_2(\theta), c_1(\theta)$, $c_2(\theta)$ such that $a_i > 0, b_i^2 - 4a_ic_i < 0$ and $d(a_i(\theta)x^2 + b_i(\theta)xy + c_1(\theta)^2)(v_i) < 0$ for $x^2 + y^2 > 0$, small .

This reads:

$$(a_i' - b_i)x^2 + (b_i' - 2c_i)xy + c_i'y^2 + 2a_i(\theta)\left(x + \frac{b_i}{2a_i}\right)\delta x_i < 0.$$

Set

$$X = x + \frac{b_i}{2a_i}y \qquad Y = y$$

so that

$$x = X - \frac{b_i}{2a_i}Y.$$

The above equation rereads:

$$(a_i' - b_i)(X - \frac{b_i}{2a_i}Y)^2 + (b_i' - 2c_i)(X - \frac{b_i}{2a_i}Y)Y + c_i'Y^2 + 2a_i(\theta)X\delta x_i < 0.$$

Setting

$$\delta x_i = A_i(\theta)X + B_i(\theta)Y + \quad \text{higher order},$$

we find

$$\begin{aligned}
&(a_i' - b_i + 2a_iA_i)X^2 + (b_i' - 2c_i + 2a_iB_i - \frac{b_i}{a_i}(a_i' - b_i))XY < 0\\
&(\frac{b_i^2}{4a_i^2}(a_i' - b_i) + c_i' - \frac{b_i}{2a_i}(b_i' - 2c_i))Y^2 < 0,
\end{aligned}$$

i.e.,

$$\begin{cases}
(b_i' - 2c_i + 2a_iB_i - \frac{b_i}{a_i}(a_i' - b_i))^2 - 4(c_i' - \frac{b_i}{2a_i}(b_i' - 2c_i) + (\frac{\ell_i}{2a_i})^2(a_i' - b_i))\\
(a_i' - b_i + 2a_iA_i) < 0\\
2a_iA_i + a_i' - b_i < 0.
\end{cases}$$

This implies:

$$\left(\frac{b_i}{2a_i}\right)^2(a_i' - b_i) + c_i' < \frac{b_i}{2a_i}(b_i' - 2c_i).$$

Furthermore,

$$2a_iA_i + a_i' - b_i \quad \text{must be sufficiently negative,}$$

once B_i is given.

This last condition is easy to satisfy as we build a convex-combination of v_1 and v_2 in $\ker\alpha$. $\mathcal{O}$ should be an attractive orbit for the convex-combination. We thus consider the condition:

$$\left(\frac{b_i}{2a_i}\right)^2(a_i' - b_i) + c_i' < \frac{b_i}{2a_i}(b_i' - 2c_i)$$

i.e.,

$$c_i' + \frac{b_i}{a_i}\left(c_i - \frac{b_i^2}{4a_i}\right) < \left(\frac{b_i b_i'}{2a_i}\right) - \left(\frac{b_i}{2a_i}\right)^2 a_i' = \left(\frac{b_i^2}{4a_i}\right)'.$$

Setting

$$c_i - \frac{b_i^2}{4a_i} = \psi,$$

we derive:

$$\psi' < -\frac{b_i}{a_i}\psi.$$

Thus we need to find, given a_i, b_i, a positive periodic function ψ such that

$$\frac{\psi'}{\psi} < -\frac{b_i}{a_i}.$$

The only conditiion is therefore to have:

$$\int_0^1 \frac{b_i}{a_i} < 0.$$

Assuming such a condition is fulfilled (it has to be for $i = 1, 2$, but we are considering more general a_i, b_i), ψ, i.e., c_i is easy to build.

There, v_1 and v_2 can be deformed one onto the other among vector-fields of $\ker\alpha$ for which $\mathcal{O}$ is attractive. q.e.d.

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