RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2007, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. Prove that the series

$$f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly on every compact subset of $\mathbb{R}$, but does not converge absolutely. Prove or disprove that $f(x)$ is continuous.

2. A function is holomorphic in the unit disk $\{z||z| < 1\}$ and satisfies the identity

$$f\left(\frac{i}{n}\right) = \frac{100}{n^4} \quad \text{for all } n = 1, 2, \ldots,$$

where i is a square root of -1. Find a formula for $f(z)$.

3. Let V be a real vector space of finite dimension n. Suppose that $L : V \to V$ is a linear transformation satisfying $L^2 + I_n = 0$, where I_n is the identity transformation of V. Show that n is even: $n = 2m$ for some $m \in \mathbb{Z}^+$, and that there is a basis for V in which the matrix representation for L is

$$\begin{pmatrix} O_m & I_m \\ -I_m & O_m \end{pmatrix},$$

where O_m and I_m are the $m \times m$ zero and identity matrix respectively.

The exam continues on next page

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let $f \in L^1[0,1]$, $E_n \subset [0,1]$ be measurable subsets, and

$$g_n(x) = \int_0^x \chi_{E_n}(t) f(t) dt.$$

Prove that there exists $\{n_k\}$ and a continuous function $g(x)$ on $[0,1]$ such that $g_{n_k}(x)$ converges uniformly to $g(x)$ on $[0,1]$.

5. A holomorphic function $f(z)$ in the punctured plane $\mathbf{C} \setminus \{0\}$ satisfies

$$|f(z)| \leq c|z|^{-\frac{1}{2}} + a|z|^{3/2}.$$

Prove that $f(z)$ is a linear polynomial.

6. Let H be a finite simple group acting transitively on a set of N objects. Show that either $N = 1$ or the order of H is at most $N!$.

7. Show that an interval in $\mathbf{R}^1$ is connected. Furthermore, show that if X is a connected subset of $\mathbf{R}^1$, then X is an interval.

The exam continues on next page

8. Suppose that $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$.

(a) Prove that $f(x-y)g(y)$ is measurable in $\mathbb{R}^2$, and for all $x \in \mathbb{R}$, $y \mapsto f(x-y)g(y)$ is measurable in $y \in \mathbb{R}$.

(b) Prove that for almost all $x \in \mathbb{R}$, $y \mapsto f(x-y)g(y)$ is in $L^1(\mathbb{R})$, and

$$\left[\int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-y)g(y)dy \right|^p dx\right]^{\frac{1}{p}} \leq ||f||_{L^p(\mathbb{R})}||g||_{L^1(\mathbb{R})}.$$

9. Find all groups of order 2006 (Hint: the prime factorization is $2006 = 2 \cdot 17 \cdot 59$).

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2007, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. Let Ω be a bounded domain in $\mathbb{R}^3$. Define for $x \in \mathbb{R}^3$

$$N(x) = \int_\Omega |x - y|^{-1} dy.$$

Prove that $N(x)$ is a C^1 function on $\mathbb{R}^3$, and

$$\frac{\partial N(x)}{\partial x^i} = \int_\Omega \frac{\partial |x-y|^{-1}}{\partial x^i} dy, \text{ for } i = 1, 2, 3.$$

Make sure to justify your differentiation under the integral sign.

2. Let f be a holomorphic function, $f : D \to D$ where $D = \{z||z| < 1\}$. Prove that if there exist $z_1,\ z_2 \in D$, such that $z_1 \neq z_2$ and such that $f(z_j) = z_j,\ j = 1, 2$, then $f(z) = z$ for all $z \in D$.

3. Let $R = \mathbf{C}[x, y]$ be the ring of complex polynomials in the variables x and y. Show that the rings $R/(x - y^2)$ and $R/(x^2 - y^2)$ are not isomorphic.

The exam continues on next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Compute

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx.$$

5. Suppose that Ω is a bounded domain in $\mathbb{R}^3$ whose boundary, $\partial\Omega$, is a C^1 hypersurface. Let $\nu(\mathbf{y}) = (\nu_1(\mathbf{y}), \nu_2(\mathbf{y}), \nu_3(\mathbf{y}))$ denote the unit exterior normal vector to $\partial\Omega$ at $\mathbf{y} \in \partial\Omega$, and $d\sigma(\mathbf{y})$ denote the area form for $\partial\Omega$.

(a) Prove that

$$\int_{\partial\Omega} \frac{\mathbf{y}\cdot\nu(\mathbf{y})}{|\mathbf{y}|^3} d\sigma(\mathbf{y}) = \begin{cases} 0, & \text{if } 0 \in \mathbb{R}^3 \setminus \bar{\Omega}; \\ 4\pi, & \text{if } 0 \in \Omega. \end{cases}$$

(b) Fix a domain Ω satisfying the assumptions above, define for $\mathbf{x} \in \mathbb{R}^3$

$$V_i(\mathbf{x}) = \int_{\partial\Omega} |\mathbf{x}-\mathbf{y}|^{-1}\nu_i(\mathbf{y})d\sigma(\mathbf{y}).$$

Prove that

$$\sum_{i=1}^{3} \frac{\partial V_i(\mathbf{x})}{\partial \mathbf{x}^i} = \begin{cases} 0, & \text{if } \mathbf{x} \in \mathbb{R}^3 \setminus \bar{\Omega}; \\ 4\pi, & \text{if } \mathbf{x} \in \Omega. \end{cases}$$

6. Let G be the group of isometries of the Euclidean plane and T be the subgroup consists of translations. Show that T is a normal subgroup of G and find the quotient group G/T.

7. Suppose (X, d) is a compact metric space and $\mathcal{U}$ is an open cover of X. Show that there is a positive number ϵ with the following property: for every point $x \in X$, there is an open set $U \in \mathcal{U}$ so that U contains the ball $B_\epsilon(x)$ of radius ϵ centered at x.

The exam continues on next page

8. Let G be a finite subgroup of the group of invertible $n \times n$ rational matrices. Show that the characteristic polynomial of any $g \in G$ has integer coefficients.

9. Let $\{f_n(z)\}$ be a sequence of holomorphic functions defined in the open unit disk $D = \{z : |z| < 1\}$. Assume Re $f_n \geq 0$. Show that either $|f_n(z)| \to \infty$ as $n \to \infty$ for all $z \in D$, or $\{f_n\}$ has a subsequence which converges uniformly on every compact subset of D.

Exam Day 2 End