Day 1 — Required

1–1. Let $\mathbf{A}$ be an $n \times n$ (real) symmetric matrix.

(a) Prove that all eigenvalues of $\mathbf{A}$ are real.

(b) Prove that two eigenvectors x and y of $\mathbf{A}$ with distinct eigenvalues λ_x and λ_y are orthogonal.

(c) If x is an eigenvector of A, find an $(n-1)$-dimensional $\mathbf{A}$-invariant subspace V of $\mathbb{R}^n$ such that $x \notin V$. *Recall that a subspace V of $\mathbb{R}^n$ is $\mathbf{A}$-invariant iff for all $v \in V$, $\mathbf{A}v \in V$.*

1–2. Find all possible values of

$$\oint_\Gamma \frac{e^{2\pi z}}{(z-1)(z-i)^2}\, dz$$

where Γ ranges over the class of simple, closed, smooth curves with $\Gamma \subset \mathbb{C} \setminus \{1, i\}$.

1–3. Let $f \in L^1(-\infty, \infty)$. Show that the functions $f(x)(\sin x)^n$, $n = 1, 2, \cdots$, are measurable and integrable on $(-\infty, \infty)$ and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)(\sin x)^n\, dx = 0.$$

Day 1 — Select 3

1–4. Let $\mathbb{F}_2$ be the field with 2 elements, and let $G = GL(3, \mathbb{F}_2)$ be the general linear group of 3×3 invertible matrices over $\mathbb{F}_2$. Show that G has an element of order 7, and find all possible minimal polynomials for elements of order 7 in G.

1–5. Let $\{f_n(z)\}$ be a sequence of holomorphic functions defined in the open unit disk $D = \{z : |z| < 1\}$. Assume $\|f_n\|_{L^2(D)} \leq 1$. Show that $\{f_n\}$ has a subsequence which converges uniformly on every compact subset of D.

1–6. Let $\{E_k\}_{k=1}^{\infty}$ be a collection of Lebesgue measurable subsets of $\mathbb{R}^n$ with $\sum_{k=1}^{\infty} |E_k| < \infty$. Show that the set consisting of all points which belong to infinitely many of the E_k has measure 0.

1–7. Let $R = \mathbb{Z} \oplus 2x\mathbb{Z}[x]$ be the subring of the ring of polynomials $\mathbb{Z}[x]$ given by

$$R = \left\{ f(x) = a_0 + 2a_1 x + 2a_2 x^2 + \cdots + 2a_n x^n : a_i \in \mathbb{Z} \right\}.$$

That is, the constant term of a polynomial in R is any integer, but the coefficients of positive powers of x are even integers. Show that R is not Noetherian.

1–8. Let $z = x + iy$ and $f(z) = x(x^2 - 3y^2) + iy(3x^2 + y^2)$. State whether each of the following is true or false and give a reason.

(a) $f'(0)$ exists.

(b) f satisfies the Cauchy-Riemann equations at $z = 0$.

(c) f is analytic (holomorphic) in a neighborhood of $z = 0$.

1–9. Let $R > 0$ and let f be an infinitely differentiable real-valued function on $\mathbb{R}^n$ with support in a ball B_R of radius R centered at the origin. *Recall that the support of a function $f(x)$ is the closure of the set of all x in the domain of f for which $f(x) \neq 0$.*

(a) Show that $|f(x)| \leq c \int_{B_R} \frac{|\nabla f(y)|}{|x-y|^{n-1}}\, dy$ for some constant c which depends on n but not on f, x or B_R. *Hint: start by applying the fundamental theorem of calculus along rays issuing from x.*

(b) Using the above, show that if $n < p < \infty$, then there is a constant $c(n,p)$ such that $|f(x)| \leq c(n,p)\, R^{1-n/p} \left(\int_{B_R} |\nabla f(y)|^p \, dy \right)^{1/p}$.

Day 2 — Required

2–1. Consider the system of equations

$$\begin{aligned} t^2 + x^3 + y^3 + z^3 &= 0 \\ t + x^2 + y^2 + z^2 &= 2 \\ t + x + y + z &= 0 \end{aligned}$$

and the point P with coordinates $(t, x, y, z) = (0, -1, 1, 0)$.

(a) Show that the above equations determine a differentiable curve which can be expressed as

$$x = x(t), \qquad y = y(t), \qquad z = z(t)$$

in some neighborhood of the point P. *Here "determine a differentiable curve" means that there is a differentiable function F with $F(t) = (x(t), y(t), z(t))$ and $F(0) = (-1, 1, 0)$.*

(b) Determine a nonzero tangent vector to the curve in (2–1.a) at the point P.

2–2. (a) Show that a compact metric space is complete.

(b) Two metrics d_1 and d_2 on a space X are said to be "equivalent" if they generate identical topologies on X. If d_1 and d_2 are equivalent on X, does completeness of (X, d_1) imply completeness of (X, d_2)? Prove it or give a counterexample.

2–3. For $n \geq 2$ let A_n denote the alternating group of even permutations of the set $\{1, 2, \ldots, n\}$. Let K be the subgroup of A_n fixing a given element, say the element 1. If H is any subgroup of A_n with $[A_n : H] = n$, show that there is an isomorphism φ of A_n with itself such that $\varphi(H) = K$.

Day 2 — Select 3

2–4. Let $\{f_n\}$ and f be Lebesgue measurable functions on $\mathbb{R}$, and let m denote Lebesgue measure on $\mathbb{R}$. Determine whether or not each of the following statements is true or false, giving a proof if it is true, a counterexample if false.

(a) If $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x, then $f_n \to f$ in measure as $n \to \infty$.

(b) If $\lim_{n\to\infty} \int_{\mathbb{R}} |f_n - f|\, dm = 0$, then $f_n \to f$ in measure as $n \to \infty$.

(c) Let f_n be integrable for each n, and let $g(x) = \liminf_{n\to\infty} f_n(x)$. Then

$$\int g(x)\, dm \le \liminf_{n\to\infty} \int f_n\, dm.$$

2–5. Find a conformal mapping of the sector

$$\frac{\pi}{6} < \arg z < \frac{\pi}{3}$$

of the complex plane onto the open disk $|z| < 1$. Explain your answer.

2–6. Let $\vec{u}\,(x, y, z)$ denote the vector field

$$\vec{u}\,(x, y, z) = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right)$$

on $\mathbb{R}^3 \setminus \{(0,0,0)\}$, and let $\mathcal{D}$ denote the annular region

$$1 < x^2 + y^2 + (z-2)^2 < 9.$$

(a) Compute the divergence of $\vec{u}$.

(b) Evaluate $\int\int_{\partial\mathcal{D}} \vec{u} \cdot \vec{n}\, ds$, where ds denotes the element of surface area of $\partial\mathcal{D}$ and $\vec{n}$ denotes the unit outer normal to $\partial\mathcal{D}$.

2–7. Suppose that $f_n(x)$ is a continuous real-valued function on $[a, b]$ for $n = 1, 2, \ldots$, and that for each $x \in [a, b]$, $\{f_n(x)\}$ is a bounded sequence. Prove that there is a subinterval of $[a, b]$ where $\{f_n(x)\}$ is uniformly bounded.

2–8. Let R be the subring of $\mathbb{C}$ generated by the integers $\mathbb{Z}$ and $\sqrt{-5} = i\sqrt{5}$, that is,

$$R = \mathbb{Z}\left[\sqrt{-5}\right].$$

(a) Show that the element $x = 2 + \sqrt{-5}$ is not a product of two nonunits in R, that is, if $2 + \sqrt{-5} = rs$ for some r and s in R, then either r or s has an inverse in R.

(b) Show that the ring R/xR is not an integral domain, where $x = 2 + \sqrt{-5}$.

(c) Show that, for m and n nonzero integers, $m + 0\sqrt{-5}$ and $n + 0\sqrt{-5}$ have a greatest common divisor in R.

2–9. The exponential of a complex matrix $\mathbf{A}$ is the power series

$$\exp(\mathbf{A}) = \sum_{i=0}^{\infty} \frac{(\mathbf{A})^i}{i!}.$$

You may use the properties that if $\mathbf{A}$ and $\mathbf{B}$ are commuting matrices, then $\exp(\mathbf{A}+\mathbf{B}) = \exp(\mathbf{A}) \cdot \exp(\mathbf{B})$, but this in general fails if $\mathbf{AB} \neq \mathbf{BA}$.

(a) Show that any matrix $\mathbf{A}$ is similar to a matrix $\mathbf{B}$ which is a sum $\mathbf{B} = \mathbf{D} + \mathbf{N}$ of two commuting matrices $\mathbf{D}$ and $\mathbf{N}$ with $\mathbf{D}$ diagonal and $\mathbf{N}$ nilpotent, that is, $\mathbf{N}$ has some power equal to 0.

(b) It is a fact that the exponential $\exp(\mathbf{A}t)$ has columns which form a basis for the solution space of the system of differential equations $d\vec{\mathbf{X}} \big/ dt = \mathbf{A}\vec{\mathbf{X}}$. Use this to find a basis for the solution space of the system

$$\begin{bmatrix} dx_1/\,dt \\ dx_2/\,dt \\ dx_3/\,dt \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$