RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2008, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts.

- Answer all three of the questions in Part I (numbered 1–3)
- Answer three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate (as directed below) which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam:

- Be sure your ID is on each book that you are submitting
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

First Day—Part I: Answer each of the following three questions

1. Use the Residue Theorem to compute $\displaystyle\int_{-\infty}^{\infty} \frac{x}{\left(x^2 - 2x + 5\right)^2}\, dx$. Describe clearly the contour of integration and the residue computation. If you claim that the limit of some integral occurring in your calculation is zero, explain carefully why this is so.

2. Assume that $f_n \leq g_n \leq h_n$ are real-valued Lebesgue integrable functions converging almost everywhere to Lebesgue integrable functions f, g, and h respectively. Assume that $\lim_n \int f_n\, dx = \int f\, dx$ and $\lim_n \int h_n\, dx = \int h\, dx$. Prove that $\lim_n \int g_n\, dx = \int g\, dx$. (Hint: The monotone convergence theorem will not work. Look at $h_n - g_n$ and $g_n - f_n$.)

3. Suppose that G is a group of order $p^2 q$ for some primes $p \neq q$. Prove that G has a normal Sylow subgroup. (You may quote the Sylow theorems without proof.)

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. (a) Prove that there is no function f holomorphic in a neighborhood of 0 having the property that $f\left(\frac{1}{n}\right) = \frac{1}{n^n}$ for all sufficiently large positive integers n.

(b) Prove that there is no entire function f so that $f\left(1+\frac{1}{n}\right) = \log\left(1+\frac{1}{n}\right)$ for all sufficiently large positive integers n.

5. Suppose f is an entire function.

(a) Define $\|f\|_{R,\infty}$ to be $\sup_{|z|\le R} |f(z)|$. Explain why $\|f\|_{R,\infty}$ is an increasing function of R, and show that if there exist R and $\widetilde{R}$ with $R < \widetilde{R}$ and $\|f\|_{R,\infty} = \|f\|_{\widetilde{R},\infty}$, then f must be constant.

(b) Define $\|f\|_{R,2}$ to be $\sqrt{\int\int_{|z|\le R} |f(z)|^2\,dx\,dy}$. Explain why $\|f\|_{R,2}$ is an increasing function of R. For which f's are there R and $\widetilde{R}$ with $R < \widetilde{R}$ and $\|f\|_{R,2} = \|f\|_{\widetilde{R},2}$?

6. Let $f \in L^1[0,1]$, the space of all L^1 integrable functions on $[0,1]$. Prove that

$$\lim_{t\to 0^+} \int_0^{1-t} |f(x+t) - f(x)|dx = 0$$

7. Given a linearly independent set of vectors $\{v_1, v_2, \ldots, v_k\}$ in $\mathbb{R}^n$, let A be the $k \times k$ matrix of dot products, i.e., $a_{ij} = v_i \cdot v_j$. Prove that A is invertible. (Be sure that your proof covers the case $k = n$.)

8. Prove that the limit $\lim_{n\to\infty}[\sum_{k=1}^n 1/k - \ln(n)]$ exists.

9. Let $M_2(\mathbb{Z})$ be the ring of 2×2 matrices with entries in $\mathbb{Z}$, and let $GL_2(\mathbb{Z})$ be the group of invertible elements of $M_2(\mathbb{Z})$ (i.e., matrices which have multiplicative inverses in $M_2(\mathbb{Z})$). Given A in $M_2(\mathbb{Z})$, prove that there exist B, C in $GL_2(\mathbb{Z})$ and integers d_1 and d_2 with d_1 dividing d_2 such that BAC is a diagonal matrix with entries d_1 and d_2. (In proving this, do not simply quote a general theorem that includes this result as a special case.)

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2008, Day 2

This examination will be given in two three-hour sessions, today's being the second part.
At each session the examination will have two parts.

- Answer all three of the questions in Part I (numbered 1–3)
- Answer three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate (as directed below) which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam:

- Be sure your ID is on each book that you are submitting
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

Second Day—Part I: Answer each of the following three questions

1. Let A_1, A_2, A_3, ... be a sequence of disjoint Lebesgue measurable sets in [0,1]. Show that for any subset $E \subset [0,1]$ (E can be non-measurable):

$$m_e\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m_e(E \cap A_i),$$

where $m_e(\ldots)$ is the outer measure.

2. Suppose that A and B are n by n matrices with entries in a field $\mathbb{F}$, that are each diagonalizable over $\mathbb{F}$. Prove that A and B commute if and only if there is a basis for $\mathbb{F}^n$ whose elements are eigenvectors for both A and B.

3. Suppose K_n, n=1,2,...., are compact and connected sets in $\mathbb{R}^m$ so that $X = \cap_{n=1}^{\infty} K_n \neq \emptyset$.
(a) Show that X is compact.
(b) Show that $\cup_{n=1}^{\infty} K_n$ is connected.

The exam continues on the next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. (a) Find a harmonic function u defined in a domain $D \subset C$ so that u is not the real part of any analytic function defined on D.
(b) Suppose that u is a harmonic function defined on a domain D so that u has a local maximum at a point p in D. Show that u is a constant.

5. Let $f_n : \; [0,1] \to [0,\infty)$ be a positive Lebesgue integrable function for $n = 1, 2, 3, \ldots$. Assume that

$$\int_0^1 f_n(x)\,dx = 1 \text{ and } \int_{1/n}^1 f_n(x)\,dx < 1/n \text{ for all } n.$$

Let

$$g(x) = \sup\{f_n(x) : \; n \geq 1\}.$$

Prove that

$$\int_0^1 g(x)\,dx = \infty.$$

6.Suppose U is an open subset of $\mathbb{C}$ and $\{f_n\}$ is a sequence of holomorphic functions which converge uniformly on compact subsets of U to a function f. Also suppose that f is <u>not</u> identically 0, and $f(w) = 0$ for some $w \in U$. Prove that there is an integer n_0 and a sequence $\{z_n : n \geq n_0\}$ of points in U such that $f_n(z_n) = 0$ for each $n \geq n_0$, and $\lim_{n\to\infty} z_n = w$.

7. Let A, B be $n \times n$ complex matrices. Show that A is similar to B if and only if the matrices $(\lambda I - A)^k$ and $(\lambda I - B)^k$ have the same rank for each complex number λ and each $k = 1, 2, 3, \ldots$.

8. Suppose f is continuous on $[0,1]$, and the function Tf is defined by

$$(Tf)(y) = y \int_0^1 \frac{f(xy)}{5 + x^3}\,dx\,.$$

(a) Prove that if f is continuous, then Tf is also continuous on $[0,1]$.

(b) Prove that if $\{f_n\}$ is any sequence of continuous functions which is uniformly bounded, then the sequence $\{T(f_n)\}$ of continuous functions has a subsequence which converges uniformly.

9. Let R be the following subring of the complex numbers:

$$R = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}.$$

Show that R is a principal ideal domain.

Exam Day 2 End