Topics for oral qualifying exam for Waldeck Schutzer, Spring 2001

July 26, 2001

1 Special Topics

1.1 The Littlewood-Richardson Rule

The product of two Schur functions and Skew Schur functions , Tableaux with lattice permutation property, statement of the Rule, the Robinson bijection and the proof of the Rule.

1.2 The Weyl Character Formula

Statement of the WCF and proof for the special case of GL_n.

2 Lie Algebras

2.1 Basic concepts

The notion of Lie algebra. Linear Lie algebras. Lie algebras of derivations. Abstract Lie algebras. Ideals. Homomorphisms and representations. Automorphisms. Solvability. Nilpotency. Engel's theorem.

2.2 Semisimple Lie algebras

Lie's theorem. Jordan-Chevalley decomposition. Cartan's criterion. Criterion for semisimplicity. Simple ideals. Inner derivations. Abstract Jordan decomposition. Modules. Casimir element of a representation. Weyl's theorem. Preservation of Jordan decomposition. Weights and maximal vectors. Classification of irreducible $\mathfrak{sl}_n\mathbb{C}$-modules. Maximal toral subalgebras and roots. Centralizer of H. Orthogonality, Integrality and Rationality properties.

2.3 Root Systems

Reflections in a euclidean plane. Root systems. Pairs of roots. Bases and Weyl chambers. Simple roots. The Weyl group. Irreducible root systems.

Cartan matrix. Coxeter graphs and Dynkin diagrams. Irreducible components. Classification theorem. Abstract theory of weights.

2.4 Existence Theorem

Universal enveloping algebras. Tensor and symmetric algebras. Construction of $\mathfrak{U}(L)$. PBW theorem and its consequences. Free Lie algebras. Generators and relations. Relations satisfied by L. Serre's theorem. Existence and uniqueness theorems. The simple Lie algebras. Criterion for semisimplicity. The classical algebras.

3 Symmetric Functions

3.1 Partitions

The notion of partition, partition diagrams, conjugate of a partition, skew diagrams and tableaux, addition and multiplication of partitions, the natural ordering, raising operators, the hook-length and the content.

3.2 The ring of symmetric functions

The ring Λ of symmetric functions, monomial, elementary, complete and forgotten symmetric functions, the generating functions $E(t)$, $H(t)$, and $P(t)$, the involution ω, power sums and the ring of symmetric functions $\Lambda_{\mathbb{Q}}$ over the rationals, a few specializations.

3.3 Schur functions

Definition of Schur polynomial, a determinantal identity, Schur functions

3.4 Orthogonality

Cauchy formula, an ω-invariant inner product in Λ, orthogonal and orthonormal bases.

3.5 Skew Schur functions

Definition of Skew Schur functions, A determinantal identity, the Kostka numbers.

3.6 Transition matrices

The notion of transition matrix and the transition matrices corresponding to the bases m, e, h, f and s

3.7 The characters of symmetric groups

The representation ring R^n of the symmetric group S_n, the ring $R = \oplus_{n\geq 0} R^n$, the characteristic map from R to Λ. The irreducible characters and the character table of S_n. Frobenius formula and Young's rule. The internal product.

References

[1] William Fulton & Joe Harris, *Representation Theory - A First Course*, GTM 129, Springer-Verlag, 1991.

[2] James E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Third Printing, GTM 9, Springer-Verlag, 1980.

[3] Nathan Jacobson, *Lie Algebras*, Dover Publications, Inc, 1979.

[4] Ian G. MacDonald, *Symmetric Functions and Hall Polynomials*, Second Edition, Reprint, Oxford Science Publications, 1998.