Syllabus for Oral Examination

Richard Mikula

Linear Functional Analysis

Banach spaces

1. Bounded Linear Transformation Theorem
2. Hahn-Banach Theorem and its applications
3. Dual space of a Banach space
4. Canonical embedding of a Banach space X into its double dual, and reflexive spaces
5. Quotient spaces
6. Baire Category Theorem
7. Banach-Steinhaus Theorem
8. Open Mapping Theorem and Inverse Mapping Theorem
9. Closed Graph Theorem

Topology

1. Compactness
   - Bolzano-Weierstrass Theorem
   - Tychonoff's Theorem
   - Banach-Alaoglu Theorem
2. Urysohn's Lemma
3. Norm and weak topologies on a Banach Space X
4. The weak* topology on X^*
5. The uniform, strong, and weak operator topologies

Hilbert spaces

1. Pythagorean identity
2. Parallelogram identity
3. Cauchy-Schwartz inequality
4. Polarization identity
5. Continuity of the inner product
6. Orthonormal sets and bases of Hilbert spaces
7. Bessel's inequality and Parseval's Theorem
8. Projection Theorem and direct sum decomposition
9. Riesz Representation Theorem

10. Hilbert adjoint operator

General spectral theory of an operator $T \in L(X, X)$

1. Resolvent set, resolvent operator and the spectrum
2. Power series representation of the resolvent operator
3. Spectral Radius Theorem
4. Non-emptiness of the spectrum
5. Spectral Mapping Theorem

Compact operators

1. Operators with finite dimensional domain and range
2. Integral operators
3. Norm limit of compact operators
4. Spectrum of a compact operator
5. The Fredholm Alterative

Bounded self-adjoint linear operators

1. The spectrum of a self-adjoint operator
2. Positive operators
3. Projection operators
4. The spectral representation of a self-adjoint operator

Unbounded operators

1. Hellinger-Toeplitz Theorem

Elliptic Partial Differential Equations

Laplace's equation

1. The fundamental solution
2. Solving Poisson's equation in the Hölder continuous case
3. Mean-value inequalities
4. The maximum principle
5. Harnack inequality
6. Green's representation formula and the Green's function for the unit ball and half space
7. Converse of mean-value formula for continuous functions

8. Regularity of harmonic functions
9. The Dirichlet problem and Perron's method

Sobolev spaces

1. Interior and global approximations by smooth functions
2. Extensions
3. Traces
4. Sobolev estimates
5. Gagliardo-Nirenberg-Sobolev Inequality
6. Poicaré's Inequality
7. Morrey's Inequality
8. Rellich-Kondrachov Compactness Theorem
9. Difference quotients
10. Rademacher's Theorem

Second order elliptic equations

1. Lax-Milgram Theorem
2. Energy estimates for the bilinear form B associated to a second order elliptic operator L
3. Existence theorems for weak solutions
4. Eigenvalues and eigenfunctions
5. Regularity theorems: interior and boundary regularity
6. Maximum principles