# Oral Qualifying Exam Syllabus

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# I. Combinatorics, Graph Theory, and the Probabilistic Method

## 1 Combinatorics

**Basic Enumeration:** counting arguments ([K04, Chapter 1], [vLW, Chapter 13, p. 119ff]),
Stirling's rule ([K04, §2.1], [F, p. 52–54]),
binomial coefficients ([K04, Chapter 1, Chapter 2]),
inclusion-exclusion principle ([K04, Chapter 4], [vLW, Chapter 10, p. 89]),
recurrence relations and generating functions ([K04, Chapter 3], [vLW, Chapter 14, p. 129ff]),

**Set Systems:** Erdős-Ko-Rado ([vLW, p.56], [Thom, Thm 2.1.9], [Bol, p. 45, §7]),
Sperner's theorem and LYM inequality ([Bol, p. 10], [vLW, p. 54], [Thom, Appendix], [K04, Chapter 5, also p. 124]),
Kruskal-Katona ([Bol, Thm 3, p. 30], [Thom, Apdx. §2.1.5]),
Fisher's and generalized Fisher's inequalities ([Thom, Lem 1.2.1], [vLW, p. 222]),
Ray-Chaudhuri–Wilson and Frankl-Wilson ([Thom, Thm 1.2.3 & 1.2.4], [FW], [vLW, Thm 19.8, p. 222]),
Borsuk Conjecture and Kahn-Kalai counterexample ([Thom, Cor 1.3.2], [KK]),

**Lattices and Posets:** distributive lattices and geometric lattices ([Bol, p. 145], [K04, Chapter 5, Chapter 9], [Thom, §1.8], [vLW, p. 305ff]),
Birkhoff representation theorem ([AS, bottom of p. 83], [Bol, p. 146]),
Dilworth ([vLW, Thm 6.1], [K04, Thm 5.2.1]),
Möbius inversion ([vLW, Chapter 25]),
Weisner ([vLW, Thm 25.3]),
Dowling-Wilson ([vLW, Thm 25.5]).

**Correlation Inequalities:** Harris-Kleitman ([Bol, p. 143], [Thom, §1.8, Thm 1.8.1]),
Fortuin-Kasteleyn-Ginibre (FKG inequality) ([Bol, p. 147], [Thom, Thm 1.8.4], [AS, p. 84], [TV, p. 34]),
Ahlswede-Daykin four functions theorem ([Bol, Chapter 19], [Thom, Thm 1.8.2], [AS, p.82]),
application to Shepp's $xyz$ inequality ([S], [AS, §6.4, p. 88]).

**Ramsey Theory:** Ramsey's theorem ([vLW, p. 28], [Thom, Apdx.], [GRS]),
infinite Ramsey ([Bol, Chapter 20], [GRS]),
probabilistic lower bounds ([AS, p. 16, p. 25, p. 67], [vLW, p. 30]),
Hales-Jewett, van der Waerden, Gallai's theorem ([GRS, Chapter 2], [Gow, Chapter 1]),
statement of Szemerédi's theorem ([TV], [Gow]).

## 2 Graph Theory

**Matching:** König ([D, Thm 2.1.1], [K04]),
Hall ([D, Thm 2.1.2], [Bol, p. 5–6], [K04], [vLW, Chapter 5]),
Tutte's 1-factor theorem ([D, Thm 2.2.1], [K04]).

**Connectivity:** structure of 2-connected graphs ([D, §3.1]),
Menger ([D, §3.3]),
Max-Flow-Min-Cut ([vLW, p. 64], [D, p. 143]),
Kruskal's algorithm (any standard algorithms book).

**Planarity:** Euler's formula ([D, Thm 4.2.9]),
Kuratowski's theorem ([D, §4.4]),
Wagner's theorem ([D, Thm 4.4.6]).

**Coloring:** chromatic number and edge chromatic number ([D]),
5 color theorem ([D, Prop 5.1.2]),
Brooks ([vLW, Chapter 3], [D, Thm 5.2.4]),
König edge coloring theorem ([D, p. 119, Prop 5.3.1]),
Vizing's Theorem ([D, Thm 5.3.2]),
perfect graphs ([D]),
Lovász's proof of weak perfect graph conjecture ([D, Thm 5.5.4]).

**Extremal Problems:** Turán ([D, p. 165], [AS, p. 91]),
statement of regularity lemma ([D, p. 176]),
Erdős-Stone ([D, Thm 7.1.2, cf p. 186–187]).

## 3 Probabilistic Methods

**Basics:** probability spaces and random variables ([W]),
linearity of expectation ([AS, Chapter 2.1]),
Bonferroni inequalities ([K04]),
Normal, Binomial, and Poisson distributions ([W], [K05]),
conditional probability and law of total probability ([W]),
Chernoff bound ([AS, Apdx. A], [TV, p. 24, §1.3], [K05]).

**Second Moment Method:** general procedure ([JLR, §3.1], [AS, Chapter 4], [TV, §1.2]),
Chebyschev's inequality ([AS, p. 41], [TV, p. 19, §1.2]),
application to threshold function for having a certain graph as a subgraph ([JLR, Chapter 3, p. 55ff], [AS, Chapter 4], [K05]).

**Lovász Local Lemma:** symmetric and general versions ([AS, p. 64–65], [TV, §1.5], [K05]),

applications to Latin transversals ([AS, §5.6],[K05]).

**Poisson Paradigm:** Janson inequalities ([AS, §8.1], [TV, §1.6]),
application to number of triangles in $G_{n,p}$ ([AS, §10.1]),
Brun's sieve ([AS, §8.3], [K05], [JLR, Coro 6.8]),
application to number of isolated vertices in $G_{n,p}$ ([K05], [JLR, Coro 3.31] ).

**Random Graphs:** $G_{n,p}$ versus $G_{n,M}$ ([JLR, p. 2]),
monotone properties ([AS, Chapter 10, §4.4], [JLR, p. 12]),
existence of threshold functions ([JLR, Thm 1.24, §1.5], [AS, p. 156–157]),
connectedness (Erdős-Renyi) ([K05]),
probabilistic refutation of Hajós' conjecture ([K05], [Cat]).

**Martingales and Tight Concentration:** Azuma's inequality ([AS, §7.2], [JLR, p. 37ff], [K05]),
edge and vertex exposure ([AS, §7.1], [K05], [JLR, p. 39]),
application to concentration of chromatic number ([AS, §7.3]),
Talagrand's inequality ([JLR, p. 39ff], [AS, §7.5,§7.6], [V05]),
comparing Talagrand's and Azuma's inequalities ([JLR, p. 41–43], [AS, §7.7], [V05]),
application to independence number of $G_{n,1/2}$ ([JLR, p. 43]).

# II. Additive and Combinatorial Number Theory

**Structure of sumsets and applications:** basic definitions and results ([V05], [Gow], [TV, p. 6–9,§2.1,§2.2]),
Ruzsa distance and additive energy ([V05], [TV, §2.3]),
theorem showing $|r_1A - r_2A + r_3B - r_4B|$ is "small" when $|A + B|$ is "small" ([V05]),
statement of Balog-Szemerédi-Gowers Theorem ([Gow, §6.4], [TV, §2.5]).

**Geometry of Numbers:** basic results for a lattices and convex bodies in $\mathbb{R}^d$ ([V05], [Gow], [TV, Chapter 3]),
John's Theorem ([TV, Thm 3.13]),
Ruzsa's Covering Lemma ([TV, Lem 3.14]),
Volume Packing Lemma ([TV, Lem 3.24]),
Blichfeldt's lemma ([Gow], [TV, p. 153]),
Minkowski's first theorem ([V05], [Gow], [TV, p. 153]),
Minkowski's second theorem ([V05], [Gow], [TV, p. 153]),
Discrete John's Theorem ([TV, Lem 3.36]).

**Generalized Arithmetic Progressions (GAPs):** definitions of proper and non-proper GAPs ([V05], [Gow], [TV, §3.2, and p. 12]),
containing lemmas ([V05], [TV, §3.6, Thm 3.38 & Thm 3.40]).

**Freiman's Theorem:** Freiman homomorphism ([Gow], [TV2357] (which is [TV, §5.3]), [Gran]),
Freiman's Cube Lemma ([TV2357, p.129, Coro 3.19] (which is [TV, §5.2]), [V06]),
Ruzsa's lemma ([Gow], [TV2357, Lem 3.26]),
Freiman's Theorem ([Gow],[TV2357, Thm 3.32]).

**Discrete Fourier Analysis and the Littlewood-Offord problem:** Definition $X_{\mathbf{v}}^{(\mu)}$ and basic properties of its Fourier representation ([V05], [TV2357, §4.2, page 168ff] (which is [TV, §7.2])),
Halász-type concentration inequality ([V05], [TV2357, §4.2, page 172ff] (which is [TV, §7.2])),
basic Littlewood-Offord results ([V05], [TV2357, §4.1] (which is [TV, §7.1]), [Bol, p. 19]),
stronger Littlewood-Offord result with Fourier analysis ([V05], [TV2357, §4.2] (which is [TV, §7.2])).

**Note:** Some of the references are to typed notes ([Gow], [K04], [Thom], and [V05] (partially)), or handwritten notes ([Gran], [V05] (partially), [V06], [K05]). Also note that [TV] and [TV2357] reference *preliminary* drafts of a book, so page and section numbers may differ from the published version (which is due out in 2006).

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