Math 574, Lecture 2

Continued Fractions of Numbers, II

Spring 2004

1. History

This organization of the study of continued fractions can be traced to an article of Euler in 1737. The theorem that every quadratic irrational has a continued fraction that is eventually periodic was first proved by Lagrange in 1770. The version given by Gauss in 1801 remains the standard treatment of the subject, except for the characterization of inverse periods by Galois in 1828.

The book of Perron is the standard for both the arithmetic and analytic theory of continued fraction, and it includes references to many of the original articles.

Weil notes that these original sources are widely available in the collected works of their authors (although I did not find a copy of works of Lagrange in a search of the Rutgers library catalog), but he also gives a detailed description of many of the methods developed in the papers that he cites. In addition to the publications, there are letters that have been collected and published that give insight into the development of these results, and Weil includes these references as well.

Here are the references (including speculation, based on other citations, about where they may be found in collected works that have not been examined.

L. Euler, "De fractionibus continuis", *C. Pet.*, 1737 (*Opera Omnia*, I-14, 187–215).

J. L. Lagrange, "Additions au mémoire sur la resolution des équations numeriques", *Mém. Berl.* **24**, 1770 (*Œvres*, VII, 125–127)

C. F. Gauss, *Disquisitiones Arithmeticae* (English translation by Arthur A. Clarke), Springer, 1986.

E. Galois, "Démonstration d'un théorème sur les fractions contiues périodiques", *Annales de mathématiques pures et appliques* **19** (1828-29) (also in *Œvres*)

O. Perron, *Die Lehre von den Kettenbrüchen*, Chelsea reprint of 1924 edition.

A. Weil, *Number Theory: An approach through history; From Hammurapi to Legendre*, Birkhäuser, 1984.

1. Reduction.

The modern definition of a **reduced quadratic irrational** $\xi = (a + \sqrt{D})/c$ appears in Gauss' book (Article 183). It requires $\xi > 1$ and the conjugate $\xi^* = (a - \sqrt{D})/c$ to be between -1 and 0.

The steps of the continued fraction of ξ are rational functions with rational coefficients, so their application to ξ^* will give the conjugate of their application to ξ. These steps are determined to assure that all $\xi_k > 1$, so we need to show the $-1 < \xi_k^* < 0$ for large enough k. First note that, if this is true for sone k, it is also true for $k+1$. The continued fraction step involves subtracting $a_k = \lfloor \xi_k \rfloor$, which moves ξ_k^* below -1; and then inverting, which moves it back to the interval $(-1, 0)$.

Indeed, if there are at least two integers greater than ξ^* and less than ξ, the next ξ^* will be in $(-1, 0)$ by the above argument.

If there is one integer greater than ξ^* and less than ξ, when ξ is translated into $(0, 1)$, ξ^* falls in $(-1, 0)$. Then, -1, 0, and 1 are all between the next ξ and the next ξ^*. As noted above, we get a reduced pair of conjugates in **one more step**.

If there is one integer between ξ and ξ^*, but $\xi < \xi^*$, one continued fraction step gives $\xi^* < \xi$ with at least one integer between them. We have seen how these soon become reduced.

If there are **no** integers between ξ and ξ^*, the continued fraction step translates **both** into $(0, 1)$ and then inverts. This increases the difference between the conjugates. Indeed, we have seen that there is an expansion by a scale factor greater than 1 in two such steps. Thus, a finite number of steps will be sure to lead to conjugates whose difference is greater than 1. Two such numbers must have an integer between them, so they soon become reduced.

We have shown that every quadratic irrational leads to a reduced pair of conjugates in a finite number of steps of the continued fraction, and all subsequent steps give reduced pairs.

2. Reversing steps.

If the pair (ξ, ξ^*) is reduced, so is $(-1/\xi^*, -1/\xi)$ and if $a = \lfloor -1/\xi^* \rfloor$, $-a-1 < 1/\xi^* < -a$, but $1/\xi^*$ is where the previous ξ^* wound up after subtracting $\lfloor \xi \rfloor$. Since it would have started in $(-1, 0)$ if the previous pair were reduced, this shows that a is also the greatest integer in the previous ξ if that is reduced. This shows that **every** reduced pair appears in a purely periodic continued fraction.

This investigation and its interpretation seems to have been noticed first by Galois. In addition to showing that this definition of **reduced pair** is an **exact characterization** of numbers appearing in a period, it shows that reversing the period gives the negative reciprocal of the conjugate.

This admits a neat description using the matrix interpretation of linear fractional transformations. The number ξ is an abbreviation of the fraction $\xi/1$, and the numerator and denominator of this fraction should be considered as the components of a vector. The period of a continued fraction leads to a matrix

$$\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

that takes this vector to a vector **representing the same fraction**. That is, to a vector **in the same direction**. This means that this vector is an **eigenvector** of the period matrix. The corresponding eigenvalue is a root of the characteristic polynomial of the matrix, which we know is $x^2 - (a+d)x + (ad - bc)$. In this case, the determinant $ad - bc = (-1)^n$, so these eigenvalues are units of the ring with discriminant $(a+d)^2 - 4(ad-bc) = (a-d)^2 + bc$. The equation whose roots are ξ and ξ^* is $cx^2 + (d-a)x - b$, which we recognize as having the same discriminant as the eigenvalue.

Since the factors of $\mathbf{P}$ are symmetric, the product in the reverse order is the transpose of $\mathbf{P}$. However, if

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

a direct calculation shows that UMU^{-1} is the transpose of the adjugate matrix of M. For nonsingular matrices, the adjugate represents the inverse linear fractional transformation, so transposing gives the inverse action on the negative reciprocal of ξ. The inverse of a unit η is $\pm\eta^*$, so a conjugate must also be taken to get the quantity belonging to the eigenvalue η.

End of Math 574, Lecture 2