Math 574, Lecture 1

Continued Fractions of Numbers, I

Spring 2004

0. Introduction.

Continued fractions have a long history. The terminating continued fraction of a rational number is closely related to the Euclidean Algorithm. This extends to a construction defined for all real numbers. With suitable conventions, this leads to a representation of real numbers that is unique for irrational numbers.

Our main concern will be with the use of continued fractions to generated good rational approximations to real numbers, but we begin with a digression into the role of continued fractions in Algebraic Number Theory.

Euler was able to show that periodicity in the steps of this construction implied that the number was the root of a quadratic equation with rational coefficients and Lagrange established the converse. Arithmetic properties of orders (rings of algebraic integers, not necessarily maximal) in real quadratic fields are closely tied to continued fractions. This will lead to a proof of a striking new theorem of Joseph Lewittes from a preprint entitled "Continued Fractions and Quadratic Irrationals".

1. Representing Real Numbers.

Before looking at the continued fraction, let us consider the familiar **decimal expansion** of real numbers.

We know how to determine decimal expansions of all real numbers, but after an **initial step** (to be described below) we have a number x with $0 \leq x < 1$. The **leading decimal digit** of x is $a = \lfloor x \rfloor$ and $x' = x - a$ is a new quantity with $0 \leq x' < 1$. The decimal expansion is produced by iterating this construction. More precisely, we start from x_1 with $0 \leq x_1 < 1$ and define sequences of integers a_i and real numbers x_i by $a_i = \lfloor x_i \rfloor$ and $x_{i+1} = x_i - a_i$. An easy induction shows that

$$0 \leq x_i < 1 \quad \text{and} \quad a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

for all $i > 0$.

Given a sequence $\langle a_i \rangle$ of decimal digits,one forms the series

$$\sum_{i=1}^{\infty} a_i \, 10^{-i}. \tag{1}$$

Since the a_i are bounded, this series converges. Indeed, it is easily shown that the limit is nonnegative and bounded by 1. It remains to show that it converges to the number x_1 used to generate the sequence. Along the way, we explain a source of some confusion, that, when all $a_i = 9$, the series converges to 1, although 1 was excluded from the numbers used to generate the sequences a_i. We shall see that it is a **fundamental property** of representations of real numbers.

Although the limit of the partial sums of a series appears to be the natural to construct a number from the sequence of a_i, it is more convenient in other cases to define numbers as the intersection of a sequence of nested intervals. For this to work, the intervals must be **compact**. Thus, our basic interval should be the **closed** interval $[0, 1]$. Prescribing a_i for $1 \leq i \leq n$ restricts the values of the sum (1) to an interval of length 10^{-n}. Since the intervals are compact, they have a nonempty intersection that is an interval, and since the lengths approach zero, there cannot be more than one point in the intersection. The selection of the next decimal digit consists of dividing the interval [0, 10] into the 10 intervals between consecutive integers in such a way that x will be contained in the interval of possible sums. Our choice guarantees that our original number x is a possible value represented by the decimal expansion, but each infinite decimal expansion represents a unique number. Two such intervals are **almost disjoint** — they are disjoint except that adjacent intervals have a single endpoint in common. We made the construction of the decimal expansion unique by selection the rightmost interval when we had a choice, and then all subsequent a_i become 0; had we chosen the other interval, all subsequent a_i would be 9.

Since intervals on the real line are **connected**, such ambiguities will arise whenever intervals are partitioned into closed subintervals. For the decimal expansion, numbers x such that $10^n x$ is an integer for some integer n have ambiguous representations.

For numbers that are not initially in the interval [0, 1], first extract the sign of x allowing restriction to positive x. Now, there will be $k > 0$ with $x \leq 10^k$. Choose such a k and form the decimal expansion of $10^{-k}x$. Insert a **decimal point** after the first k digits of this to signify that the first digit is the coefficient of 10^{k-1}. In this notation, it is conventional to omit leading zero digits to the left of the decimal point, since they contribute noting to analog of the sum (1)

2. Continued fractions of Numbers.

The continued fraction expansion is also obtained by an inductive procedure. The inductive step, starting with a real number ξ consists of setting $a = \lfloor \xi \rfloor$ and $\xi' = 1/(\xi - a)$. If ξ is an integer, this will be consider to lead to ξ' being the **special symbol** ∞ which causes the process to terminate.

This step leads to $\xi' > 1$, but we should turn this into a closed interval by adding endpoints 1 and ∞. If we start in this interval, $a = \lfloor \xi \rfloor$ is a **positive integer**. This construction leads to the terms $a_0, a_1, a_2, \ldots$, where a_0 is an arbitrary integer and the a_i for $i > 0$ are positive integers. It is customary to indicate that this sequence arises from the continued fraction algorithm by enclosing it in brackets. The a_i are separated by commas, except that the special term (which is special since it is not required to be positive) a_0 is followed by a semicolon.

The function $f(x) = n + 1/x$ for some fixed integer n maps the interval $[1, \infty]$ into the interval $[n, n+1]$ of numbers whose continued fractions begin with n. For each interval $[x_0, x_1]$, the length of its image $[f(x_1), f(x_0)]$ is $x_0^{-1} - x_1^{-1} = (x_1 - x_0)/(x_0 x_1)$. Since both x_0 and x_1 are at least 1 and one is strictly greater, the image is strictly shorter than $[x_0, x_1]$. However, more is needed to show that the intervals defined by initial segments of a continued fraction have lengths approaching zero. To do this, we consider compositions of these maps. We have

$$f(x) = n + \frac{1}{x} = \frac{nx+1}{x}$$

so this is a special case of the **linear fractional transformation**

$$\frac{ax+b}{cx+d}.$$

Composing two such functions gives

$$\frac{a'\left(\frac{ax+b}{cx+d}\right)+b'}{c'\left(\frac{ax+b}{cx+d}\right)+d'} = \frac{a'(ax+b)+b'(cx+d)}{c'(ax+b)+d'(cx+d)} = \frac{(a'a+b'c)x+(a'b+b'd)}{(c'a+d'c)x+(c'b+d'd)},$$

so it is again a linear fractional transformation. If the coefficients are written as a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then the matrix corresponding to the composition is

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

For the special functions that are used to form continued fraction, $d = 0$, $b = c = 1$ and a is a positive integer. In particular, the determinant is -1. Composing k such terms gives a linear fractional transformation represented my a matrix of determinant $(-1)^k$ whose entries are nonnegative integers. The slowest growth of these entries occurs when all $a = 1$, and then the size of the entries in the product are asymptotic to constant multiples of τ^k, where $\tau = \left(1 + \sqrt{5}\right)/2$. This shows that the lengths of intervals defined by k partial quotients have lengths less than $C\tau^{-2}$ for some absolute constant C as $k \to \infty$.

Thus, each sequence of partial quotients determines a number. The construction of the partial quotients from the number is given by dividing $[1, \infty]$ into the intervals $[n, n+1]$ and the special value ∞. Each integer greater than 1 belongs to two intervals, but other numbers belong to unique intervals. A value in $[n, n+1]$ is written as $n + 1/x$, so n corresponds to $x = \infty$ and $n+1$ corresponds to $x = 1$. The number 1 has the string of partial quotients $\langle 1, \infty \rangle$, so each integer has a terminating continued fraction.

The image of ∞ under a linear fractional transformation represented by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a/c. Since the entries are integers for the matrices appearing in the continued fraction construction, this is rational. Conversely, if a number is rational, the continued fraction step is **exactly** a step of the **Euclidean algorithm**. Since this process terminates, every rational has a terminating continued fraction expansion — indeed, exactly **two** terminating continued fraction expansions.

3. Periodic Continued fractions.

After the terminating expansions, the next thing to consider are the periodic expansions. If a continued fraction expansion of a number x is **purely periodic**, the period gives a linear fractional transformation such that

$$x = \frac{ax+b}{cx+d}.$$

This leads to equation $cx^2 + (d-a)x - b = 0$. This is a quadratic equation with integer coefficients and discriminant $(d-a)^2 + 4bc = (a+d)^2 - 4(ad-bc)$. An efficient way to calculate the matrix that is a product of the special matrices

$$M_a = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$$

is to start by writing an identity matrix. Then as you read the numbers a_i from the continues fraction **from left to right**, introduce a new column **on the left** that is a_i times the current first column plus the current second column. This works because this is the **column operation** giving the first column after multiplying by M_{a_i} while the second column is the old first column. The first two columns of this array is the desired matrix. The a_i may be written above the column that it introduces to serve as a reminder of the steps in the calculation. For example, [4, 1, 1, 1] leads to

$$\begin{array}{cccccc} & 1 & 1 & 1 & 4 & \\ \hline 14 & 9 & 5 & 4 & 1 & 0 \\ 3 & 2 & 1 & 1 & 0 & 1 \end{array}$$

and the equation

$$x = \frac{14x+9}{3x+2}$$

expands to $3x^2 - 12x - 9 = 0$. Removing the common factor of 3 in the coefficients gives $x^2 - 4x - 3 = 0$, whose roots are $2 \pm \sqrt{7}$.

Since the determinant $ad - bc = \pm 1$ in this case, this discriminant has the form $(a+d)^2 \pm 4$. Although this appears to be special — excluding numbers like 7 — we found that the coefficients of the equation may have c common factor greater than 1 even though the entries of the matrix have no common factor. Also, if the middle coefficient is even, a factor of 2 can be removed from both the numerator and denominator of the expression given by the quadratic formula. There is a factor of 4 in the discriminant that isn't seen in the usual way of writing the solution of the equation.

The calculation of the continued fraction of $2 + \sqrt{7}$ involves only numbers of the form $(a + \sqrt{7})/c$. Here are the details:

$$2 + \sqrt{7} = 4 + \left(\sqrt{7} - 2\right)$$

$$\frac{1}{\sqrt{7}-2} = \frac{\sqrt{7}+2}{3}$$

$$\frac{\sqrt{7}+2}{3} = 1 + \frac{\sqrt{7}-1}{3}$$

$$\frac{3}{\sqrt{7}-1} = \frac{\sqrt{7}+1}{2}$$

$$\frac{\sqrt{7}+1}{2} = 1 + \frac{\sqrt{7}-1}{2}$$

$$\frac{2}{\sqrt{7}-1} = \frac{\sqrt{7}+1}{3}$$

$$\frac{\sqrt{7}+1}{3} = 1 + \frac{\sqrt{7}-2}{3}$$

$$\frac{3}{\sqrt{7}-2} = \sqrt{7}+2$$

Part of reason that this form persists through the calculation is that c divides $7 - a^2$ at each stage.

4. Lagrange's Theorem.

Euler is credited with noticing that a periodic continued fraction represents the root of a quadratic equation with integer coefficients (which we observed in the last section without making much of a fuss about it). The converse was proved by Lagrange. One key to his proof is the property noted in the example. We state it as

Lemma. *If we perform one step in the calculation of the continued fraction of* $\left(a+\sqrt{D}\right)/c$ *with* $c|D-a^2$*, then the result has the same form.*

Proof. The step consists of identifying the integer q such that

$$qc < a+\sqrt{D} < (q+1)c$$

so that

$$-\sqrt{D} < a-qc < c-\sqrt{D}$$

Subtracting q and inverting gives

$$\frac{c}{\sqrt{D}+a-qc} = \frac{c\left(\sqrt{D}+qc-a\right)}{D-(qc-a)^2}$$

The denominator is $D-a^2+2aqc-q^2c^2$. This is divisible by c since $c|D-a^2$ and the remaining terms contain explicit factors of c. Remove this factor of c from numerator and denominator to give the lemma.

Although the condition that $c|D-a^2$ may seem special, it is easy to arrange for any given quadratic irrational. If numerator and denominator of $\left(a+\sqrt{D}\right)/c$ are multiplied by the integer k, we get $\left(ak+\sqrt{Dk^2}\right)/(ck)$ with the new discriminant Dk^2. For this expression, the desired condition is $ck|Dk^2-a^2k^2$, which is equivalent to $c|k\left(D-a^2\right)$. This can always be arranged. For example, $k=c$ will certainly give this conclusion.

We assume $c|D-a^2$ for the rest of the discussion.

The second main part of Lagrange's theorem is the idea of **reduction**. When we perform a continued fraction step, a is replaced by $qc-a$, which is between $\sqrt{D}$, and $\sqrt{D}-c$, and c is replaced by $\left(D-(qc-a)^2\right)/c$. If $c>2\sqrt{D}$, then $|qc-a|\le c-\sqrt{D}$ and

$$2\sqrt{D}-c < \frac{D-(qc-a)^2}{c} < \frac{D}{c} < \frac{\sqrt{D}}{2}$$

This shows that $\left|\left(D-(qc-a)^2\right)/c\right| < c$. A finite number of continued fraction steps will lead to an expression with $c<2\sqrt{D}$. After this bound on c has been attained, $|a|<\sqrt{D}$ at all subsequent steps. Only these could possibly occur in later steps, but these conditions have limited us to a **finite set** of pairs (a,c). Additional steps of the continued fraction expansion will eventually repeat one of these values. Since the continued fraction is completely determined by the number, a single repetition forces the construction to become immediately periodic.

It has now been established that every quadratic irrational has a continued fraction that is eventually periodic.

End of Math 574, Lecture 1